IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Combinatorics Test: Solutions

Combinatorics Test

1. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

   Answer: 11
   Solution: Suppose that n seniors play all three sports and that $2n$ are on the lacrosse team. Then, by the principle of inclusion-exclusion, $85 = (74 + 26 + 2n) - (17 + 18 + 13) + (n) = 100 + 2n - 48 + n = 52 + 3n$. It is easily seen that $n = 11$.

2. Compute

$$\sum_{n_{60}=0}^{2} \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1.$$

   Answer: 1953
   Solution: The given sum counts the number of non-decreasing 61-tuples of integers $(n_0, \ldots, n_{60})$ from the set $\{0, 1, 2\}$. Such 61-tuples are in one-to-one correspondence with strictly increasing 61-tuples of integers $(m_0, \ldots, m_{60})$ from the set $\{0, 1, 2, \ldots, 62\}$: simply let $m_k = n_k + k$. But the number of such $(m_0, \ldots, m_{60})$ is almost by definition $\binom{63}{61} = \binom{63}{2} = 1953$.

3. A moth starts at vertex A of a certain cube and is trying to get to vertex B, which is opposite A, in five or fewer "steps," where a step consists in traveling along an edge from one vertex to another. The moth will stop as soon as it reaches B. How many ways can the moth achieve its objective?

   Answer: 48
   Solution: Let X, Y, Z be the three directions in which the moth can intially go. We can symbolize the trajectory of the moth by a sequence of stuff from Xs, Ys, and Zs in the obvious way: whenever the moth takes a step in a direction parallel or opposite to X, we write down X, and so on.

   The moth can reach B in either exactly 3 or exactly 5 steps. A path of length 3 must be symbolized by XYZ in some order. There are $3! = 6$ such orders. A trajectory of length 5 must by symbolized by $XYZXX$, $XYZYY$, or $XYZZZ$, in some order, There are $3 \cdot \frac{5!}{3!1!1!} = 3 \cdot 20 = 60$ possibilities here. However, we must remember to subtract out those trajectories that already arrive at B by the 3rd step: there are $3 \cdot 6 = 18$ of those. The answer is thus $60 - 18 + 6 = 48$.

4. A dot is marked at each vertex of a triangle ABC. Then, 2, 3, and 7 more dots are marked on the sides AB, BC, and CA, respectively. How many triangles have their vertices at these dots?

   Answer: 357
   Solution: Altogether there are $3 + 2 + 3 + 7 = 15$ dots, and thus $\binom{15}{3} = 455$ combinations of 3 dots. Of these combinations, $\binom{2+2}{3} + \binom{2+3}{3} + \binom{2+7}{3} = 4 + 10 + 84 = 98$ do not give triangles because they are collinear (the rest do give triangles). Thus $455 - 98 = 357$ different triangles can be formed.

5. Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of today's handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbor's. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?

   Answer: 125
   Solution: Suppose that you are one of the freshmen; then there's a $6/15$ chance that you'll get one of the handouts. We may ask, given that you do get a handout, how many ways are there to distribute the rest? We need only multiply the answer to that question by $15/6$ to answer the original question.

   Going clockwise around the table from you, one might write down the sizes of the gaps between people with handouts. There are six such gaps, each of size 0–2, and the sum of their sizes must be $15 - 6 = 11$. So the gap sizes are either 1, 1, 1, 2, 2, 2 in some order, or 0, 1, 2, 2, 2, 2 in some order. In the former case, $\frac{6!}{3!3!} = 20$ orders are possible; in the latter, $\frac{6!}{1!1!4!} = 30$ are. Altogether, then, there are $20 + 30 = 50$ possibilities.

   Multiplying this by $15/6$, or $5/2$, gives 125.

6. For how many ordered triplets (a, b, c) of positive integers less than 10 is the product $a \times b \times c$ divisible by 20?

   Answer: 102
   Solution: One number must be 5. The other two must have a product divisible by 4. Either both are even, or one is divisible by 4 and the other is odd. In the former case, there are $48 = 3 \times 4 \times 4$ possibilities: 3 positions for the 5, and any of 4 even numbers to fill the other two. In the latter case, there are $54 = 3 \times 2 \times 9$ possibilites: 3 positions and 2 choices for the multiple of 4, and 9 ways to fill the other two positions using at least one 5.

7. Let n be a positive integer, and let Pushover be a game played by two players, standing squarely facing each other, pushing each other, where the first person to lose balance loses. At the HMPT, 2^{n+1} competitors, numbered 1 through 2^{n+1} clockwise, stand in a circle. They are equals in Pushover: whenever two of them face off, each has a 50% probability of victory. The tournament unfolds in $n + 1$ rounds. In each round, the referee randomly chooses one of the surviving players, and the players pair off going clockwise, starting from the chosen one. Each pair faces off in Pushover, and the losers leave the circle. What is the probability that players 1 and 2^n face each other in the last round? Express your answer in terms of n.

Answer: $\dfrac{2^n - 1}{8^n}$
Solution: At any point during this competition, we shall say that the situation is *living* if both players 1 and 2^n are still in the running. A living situation is *far* if those two players are diametrically opposite each other, and *near* otherwise, in which case (as one can check inductively) they must be just one person shy of that maximal separation. At the start of the tournament, the situation is living and near. In each of rounds 1 to n, a far situation can never become near, and a near situation can stay near or become far with equal likelihood.

In each of rounds 1 to $n-1$, a living situation has a $1/4$ probability of staying living. Therefore, at the end of round k, where $1 \le k \le n-1$, the situation is near with probability $1/8^k$, and far with probability $1/4^k - 1/8^k$. In round n, a far situation has a $1/4$ probability of staying living, whereas a near situation has only a $1/8$ probability of staying living. But if the situation is living at the beginning of the last round, it can only be far, so we can say with complete generality that, at the end of round k, where $1 \le k \le n$, the situation is living and far with probability $1/4^k - 1/8^k$. We are interested in finding the probability that the situation is living at the end of round n (and hence far); that probability is thus $\frac{1}{4^n} - \frac{1}{8^n} = \frac{2^n - 1}{8^n}$.

8. In how many ways can we enter numbers from the set $\{1, 2, 3, 4\}$ into a 4×4 array so that all of the following conditions hold?

   (a) Each row contains all four numbers.

   (b) Each column contains all four numbers.

   (c) Each "quadrant" contains all four numbers. (The quadrants are the four corner 2×2 squares.)

Answer: 288
Solution: Call a filled 4×4 array satisfying the given conditions *cool*. There are 4! possibilities for the first row; WLOG, let it be 1 2 3 4. Since each quadrant has to contain all four numbers, we have exactly four possibilities for the second row, namely:

(i) 3 4 1 2

(ii) 3 4 2 1

(iii) 4 3 1 2

(iv) 4 3 2 1

I claim that the number of cool arrays with (i) is equal to those with (iv), and that the number of cool arrays with (ii) is equal to those with (iii). Let's first consider (i) and (iv). Now, (i) is

$$\begin{matrix} 1\ 2\ 3\ 4 \\ 3\ 4\ 1\ 2 \end{matrix}$$

while (iv) is

$$\begin{matrix} 1\ 2\ 3\ 4 \\ 4\ 3\ 2\ 1 \end{matrix}$$

In (iv), switch 3 and 4 (relabeling doesn't affect the coolness of the array); then, it becomes

$$\begin{matrix} 1 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \end{matrix}$$

Now, interchange the last two columns, which also does not affect the coolness. This gives us (i). Hence, the cool arrays with (i) and the cool arrays with (iv) have a 1:1 correspondence. Using the exact same argument, we can show that the number of cool arrays with (ii) equals those with (iii).

So we only need consider cases (i) and (ii). It is easy to verify that there are four cool arrays with (i), determined precisely by, say, the first two entries of the third row; and two with (ii), determined precisely by, say the first entry of the third row. Hence, the answer is $4! \times (4+2) \times 2 = 288$.

9. Eight celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

Answer: 3507
Solution: Let the celebrities get into one or more circles so that each circle has at least three celebrities, and each celebrity shook hands precisely with his or her neighbors in the circle.

Let's consider the possible circle sizes:

- *There's one big circle with all 8 celebrities.* Depending on the ordering of the people in the circle, the fan's list can still vary. Literally speaking, there are 7! different circles 8 people can make: fix one of the people, and then there are 7 choices for the person to the right, 6 for the person after that, and so on. But this would be double-counting because, as far as the fan's list goes, it makes no difference if we "reverse" the order of all the people. Thus, there are $7!/2 = 2520$ different possible lists here.
- *5+3.* In this case there are $\binom{8}{5}$ ways to split into the two circles, $\frac{4!}{2}$ essentially different ways of ordering the 5-circle, and $\frac{2!}{2}$ ways for the 3-circle, giving a total count of $56 \cdot 12 \cdot 1 = 672$.
- *4+4.* In this case there are $\binom{8}{4}/2 = 35$ ways to split into the two circles (we divide by 2 because here, unlike in the $5+3$ case, it does not matter which circle is which), and $\frac{3!}{2} = 3$ ways of ordering each, giving a total count of $35 \cdot 3 \cdot 3 = 315$.

Adding them up, we get $2520 + 672 + 315 = 3507$.

10. Somewhere in the universe, n students are taking a 10-question math competition. Their collective performance is called *laughable* if, for some pair of questions, there exist 57 students such that either all of them answered both questions correctly or none of them answered both questions correctly. Compute the smallest n such that the performance is necessarily laughable.

Answer: 253
Solution: Let $c_{i,j}$ denote the number of students correctly answering questions i and j ($1 \leq i < j \leq 10$), and let $w_{i,j}$ denote the number of students getting both questions

wrong. An individual student answers k questions correctly and $10-k$ questions incorrectly. This student answers $\binom{k}{2}$ pairs of questions correctly and $\binom{10-k}{2}$ pairs of questions incorrectly. Now observe that

$$\binom{k}{2} + \binom{10-k}{2} = k^2 - 10k + 45 = (k-5)^2 + 20 \geq 20$$

Therefore,

$$\sum_{1 \leq i < j \leq 10} c_{i,j} + w_{i,j} \geq 20n$$

Now if the performance is not laughable, then $c_{i,j} \leq 56$ and $w_{i,j} \leq 56$ for all $1 \leq i < j \leq 10$. Observe that there are $2\binom{10}{2} = 90$ of these variables. Hence, in a boring performance,

$$20n \leq \sum_{1 \leq i < j \leq 10} c_{i,j} + w_{i,j} \leq 90 \cdot 56 = 5040$$

or $n \leq 252$. In particular this implies that if $n \geq 253$, the performance is laughable. This is the best bound because $\binom{10}{5} = 252$, and if each of 252 students correctly answers a different 5 element subset of the 10 questions, then $c_{i,j} = w_{i,j} = 56$ for all $1 \leq i < j \leq 10$.