IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Geometry Test: Solutions

1. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

   Answer: $20 + \sqrt{2}$
   Solution: Extend sides AB, CD, EF, GH to form a rectangle: let X be the intersection of lines GH and AB; Y that of AB and CD; Z that of CD and EF; and W that of EF and GH.

   As $BC = 2$, we have $BY = YC = \sqrt{2}$. As $DE = 4$, we have $DZ = ZE = 2\sqrt{2}$. As $FG = 2$, we have $FW = WG = \sqrt{2}$.

   We can compute the dimensions of the rectangle: $WX = YZ = YC + CD + DZ = 3 + 3\sqrt{2}$, and $XY = ZW = ZE + EF + FW = 2 + 3\sqrt{2}$. Thus, $HX = XA = XY - AB - BY = 1 + 2\sqrt{2}$, and so $AH = \sqrt{2}HX = 4 + \sqrt{2}$, and $GH = WX - WG - HX = 2$. The perimeter of the octagon can now be computed by adding up all its sides.

2. Suppose ABC is a scalene right triangle, and P is the point on hypotenuse $\overline{AC}$ such that $\angle ABP = 45°$. Given that $AP = 1$ and $CP = 2$, compute the area of ABC.

   Answer: $\frac{9}{5}$
   Solution: Notice that $\overline{BP}$ bisects the right angle at B. Thus, we write $AB = 2x$, $BC = x$. By the Pythagorean theorem, $5x^2 = 9$, from which the area $\frac{1}{2}(x)(2x) = x^2 = \frac{9}{5}$.

3. Let A, B, C, and D be points on a circle such that $AB = 11$ and $CD = 19$. Point P is on segment AB with $AP = 6$, and Q is on segment CD with $CQ = 7$. The line through P and Q intersects the circle at X and Y. If $PQ = 27$, find XY.

   Answer: 31
   Solution: Suppose X, P, Q, Y lie in that order. Let $PX = x$ and $QY = y$. By power of a point from P, $x \cdot (27 + y) = 30$, and by power of a point from Q, $y \cdot (27 + x) = 84$. Subtracting the first from the second, $27 \cdot (y - x) = 54$, so $y = x + 2$. Now, $x \cdot (29 + x) = 30$, and we find $x = 1, -30$. Since -30 makes no sense, we take $x = 1$ and obtain $XY = 1 + 27 + 3 = 31$.

4. Let ABC be a triangle such that $AB = 2, CA = 3$, and $BC = 4$. A semicircle with its diameter on $\overline{BC}$ is tangent to $\overline{AB}$ and $\overline{AC}$. Compute the area of the semicircle.

   Answer: $\frac{27\pi}{40}$
   Solution: Let O, D, and E be the midpoint of the diameter and the points of tangency with $\overline{AB}$ and $\overline{AC}$ respectively. Then $[ABC] = [AOB] + [AOC] = \frac{1}{2}(AB + AC)r$, where r is the radius of the semicircle. Now by Heron's formula, $[ABC] = \sqrt{\frac{9}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}} = \frac{3\sqrt{15}}{4}$. We solve for $r = \frac{3\sqrt{15}}{10}$ and compute $\frac{1}{2}\pi r^2 = \frac{27\pi}{40}$.

5. Triangle ABC has side lengths $AB = 2\sqrt{5}$, $BC = 1$, and $CA = 5$. Point D is on side AC such that $CD = 1$, and F is a point such that $BF = 2$ and $CF = 3$. Let E be the intersection of lines AB and DF. Find the area of $CDEB$.

Answer: $\frac{22}{35}$

Solution: Draw segment AF. Then notice $AF = 4$, and we have a right triangle. Now draw line CE, let it intersect AF at G. By Ceva, $FG = \dfrac{4}{3}$ and $AG = \dfrac{8}{3}$. Using mass points we find that $\dfrac{AE}{EB} = 6$ so $\dfrac{[AEF]}{[BEF]} = 6$, and since $[ABF] = 4$, $[BEF] = \dfrac{4}{7}$. It's easy to see that $[CDF] = \dfrac{1}{5}[ACF] = \dfrac{6}{5}$, so

$$[BCDE] = [CDF] - [BEF] = \frac{6}{5} - \frac{4}{7} = \frac{22}{35}.$$

6. A circle of radius t is tangent to the hypotenuse, the incircle, and one leg of an isosceles right triangle with inradius $r = 1 + \sin\frac{\pi}{8}$. Find rt.

Answer: $\dfrac{2+\sqrt{2}}{4}$

Solution: The distance between the point of tangency of the two circles and the nearest vertex of the triangle is seen to be both $r(\csc\frac{\pi}{8} - 1)$ and $t(\csc\frac{\pi}{8} + 1)$, so

$$\begin{aligned} rt &= \frac{r^2(\csc\frac{\pi}{8} - 1)}{\csc\frac{\pi}{8} + 1} = \frac{(1+\sin\frac{\pi}{8})^2(1-\sin\frac{\pi}{8})}{1+\sin\frac{\pi}{8}} = 1 - \sin^2\frac{\pi}{8} \\ &= \frac{1}{2} + \frac{1 - 2\sin^2\frac{\pi}{8}}{2} = \frac{1}{2} + \frac{\cos\frac{\pi}{4}}{2} = \frac{1}{2} + \frac{\sqrt{2}}{4} = \frac{2+\sqrt{2}}{4}. \end{aligned}$$

7. Suppose $ABCD$ is an isosceles trapezoid in which $\overline{AB} \parallel \overline{CD}$. Two mutually externally tangent circles ω_1 and ω_2 are inscribed in $ABCD$ such that ω_1 is tangent to $\overline{AB}, \overline{BC}$, and $\overline{CD}$ while ω_2 is tangent to $\overline{AB}, \overline{DA}$, and $\overline{CD}$. Given that $AB = 1, CD = 6$, compute the radius of either circle.

Answer: $\frac{3}{7}$

Solution: Let the radius of both circles be r, and let ω_1 be centered at O_1. Let ω_1 be tangent to $\overline{AB}, \overline{BC}$, and $\overline{CD}$ at P, Q, and R respectively. Then, by symmetry, $PB = \frac{1}{2} - r$ and $RC = 3 - r$. By equal tangents from B and C, $BQ = \frac{1}{2} - r$ and $QC = 3 - r$. Now, $\angle BO_1C$ is right because $m\angle O_1BC + m\angle BCO_1 = \frac{1}{2}(m\angle PBC + m\angle BCR) = 90°$. Since $\overline{O_1Q} \perp \overline{BC}$, $r^2 = O_1Q^2 = BQ \cdot QC = (\frac{1}{2} - r)(3 - r) = r^2 - \frac{7}{2}r + \frac{3}{2}$. Solving, we find $r = \frac{3}{7}$.

8. Triangle ABC has a right angle at B. Point D lies on side BC such that $3\angle BAD = \angle BAC$. Given $AC = 2$ and $CD = 1$, compute BD.

Answer: $\frac{3}{8}$

Solution: Let $BD = x$. We reflect D over AB to D'. Then $DD' = 2x$, but AD bisects CAD', so $4x = AD' = AD$. Also, $AD = \sqrt{x^2 + AB^2} = \sqrt{x^2 + AC^2 - BC^2} = \sqrt{x^2 + 4 - (x+1)^2} = \sqrt{3 - 2x}$. We have the quadratic $16x^2 = 3 - 2x$ which gives $x = 3/8$.

9. Four spheres, each of radius r, lie inside a regular tetrahedron with side length 1 such that each sphere is tangent to three faces of the tetrahedron and to the other three spheres. Find r.

Answer: $\dfrac{\sqrt{6}-1}{10}$

Solution: Let O be the center of the sphere that is tangent to the faces ABC, ABD, and BCD. Let P, Q be the feet of the perpendiculars from O to ABC and ABD respectively. Let R be the foot of the perpendicular from P to AB. Then, $OPRQ$ is a quadrilateral such that $\angle P$, $\angle Q$ are right angles and $OP = OQ = r$. Also, $\angle R$ is the dihedral angle between faces ABC and ABD, so $\cos\angle R = 1/3$. We can then compute $QR = \sqrt{2}r$, so $BR = \sqrt{6}r$. Hence, $1 = AB = 2(\sqrt{6}r) + 2r = 2r(\sqrt{6}+1)$, so $r = (\sqrt{6}-1)/10$.

10. Triangle ABC has side lengths $AB = 65$, $BC = 33$, and $AC = 56$. Find the radius of the circle tangent to sides AC and BC and to the circumcircle of triangle ABC.

Answer: 24

Solution: Let Γ be the circumcircle of triangle ABC, and let E be the center of the circle tangent to Γ and the sides AC and BC. Notice that $\angle C = 90°$ because $33^2 + 56^2 = 65^2$. Let D be the second intersection of line CE with Γ, so that D is the midpoint of the arc AB away from C. Because $\angle BCD = 45°$, one can easily calculate $CD = 89\sqrt{2}/2$. The power of E with respect to Γ is both $r(65-r)$ and $r\sqrt{2}\cdot(89\sqrt{2}/2 - r\sqrt{2}) = r(89-2r)$, so $r = 89 - 65 = 24$.