# IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

**Saturday 25 February 2006**

**Team Round A: Solutions**

## Mobotics [120]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged at the points of a certain grid (to be specified later). Our machinery consists of a fleet of identical *mowbots* (or "mobots" for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction. At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered "different" if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot starts on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

1. **[15]** For this problem, our lawn consists of a row of $n$ clumps of grass. This row runs in an east-west direction. In our formation, each mobot may be oriented toward the north, south, east, or west. One example of an allowable formation if $n = 6$ is symbolized below:

   $$\cdot \quad \cdot \leftarrow \quad \uparrow \quad \uparrow \quad \downarrow$$

   (The mobot on the third clump will move westward, mowing the first three clumps. Each of the last three clumps is mowed by a different mobot.) Here's another allowable formation for $n = 6$, considered different from the first:

   $$\cdot \quad \cdot \leftarrow \quad \uparrow \quad \uparrow \quad \rightarrow$$

   Compute the number of different allowable formations for any given $n$.

   **Solution:** Let $a$ be the number of clumps mowed by a mobot oriented to go west, and let $b$ be the number of clumps mowed by a mobot oriented to go east. (So in the two examples given, $(a, b)$ would be $(3, 0)$ and $(3, 1)$, respectively.) We may ask how many allowable formations with a given ordered pair $(a, b)$ there are, and then sum our answers up over all possible ordered pairs $(a, b)$.

   Given any particular $(a, b)$, first of all, $a$ and $b$ had better be non-negative integers summing to at most $n$. As long as that's true, there will be $n - a - b$ clumps of grass which are each mowed by a single mobot oriented to go either north or south: there are thus $2^{n-a-b}$ possibilities. So our answer is

$$\sum_{a=0}^{n}\sum_{b=0}^{n-a} 2^{n-a-b} = \sum_{a=0}^{n}(2^{n-a} + 2^{n-a-1} + \cdots + 2^0) = \sum_{a=0}^{n}(2^{n-a+1} - 1)$$
$$= -(n+1) + \sum_{a=0}^{n} 2^{n-a+1} = -(n+1) + (2^{n+1} + 2^n + 2^{n-1} + \cdots + 2^1)$$
$$= -(n+1) + (2^{n+2} - 2) = 2^{n+2} - n - 3.$$

2. **[25]** For this problem, our lawn is an $m \times n$ rectangular grid of clumps, that is, with $m$ rows running east-west and $n$ columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x < n \text{ and } 0 \leq y < m\}.$$

However, mobots are now allowed to be oriented to go either north or east only. So one allowable formation for $m = 2$, $n = 3$ might be as follows:

$$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \uparrow & \rightarrow & \cdot \end{array}$$

Prove that the number of allowable formations for given $m$ and $n$ is $\dfrac{(m+n)!}{m!\,n!}$.

**Solution:** There is a one-to-one correspondence between allowable formations and paths from $(0,0)$ to $(n,m)$ made up of $n$ moves 1 unit to the right and $m$ moves 1 unit up. The correspondence works as follows: There must be a mobot at $(0,0)$, so start the path there. If that mobot is oriented to move up, then move to the right; if that mobot is oriented to move to the right, then move up. In doing so, you will meet another mobot, upon which you can repeat the above process, until you leave the lawn. Once you leave the lawn, there will be only one way to proceed to $(n,m)$ — either keep going right, or keep going up. (For the example in the problem, our path would be $(0,0)$–$(1,0)$–$(1,1)$–$(1,2)$–$(2,2)$–$(3,2)$.)

Conversely, given any path from $(0,0)$ to $(n,m)$, we can derive back a mobot formation: place a mobot at every lattice point *of the lawn* that the path touches, and don't orient that mobot in the same direction as the path takes when leaving that point.

It is easy to check that this is indeed a one-to-one correspondence as claimed. Every path from $(0,0)$ to $(n,m)$ consists of $n$ moves to the right and $m$ moves up done in an arbitrary order, and there are precisely $\binom{m+n}{m} = \frac{(m+n)!}{m!\,n!}$ orders.

3. **[40]** In this problem, we stipulate that $m \geq n$, and the lawn is shaped differently. The clumps are now at the lattice points in a trapezoid:

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x < n \text{ and } 0 \leq y < m + 1 - n + x\},$$

As in problem 2, mobots can be set to move either north or east. For given $m$ and $n$, determine with proof the number of allowable formations.

**Solution:** For exactly the same reasons as in problem 2, we have a one-to-one correspondence between the allowable formations and paths (going 1 unit up or right at a time) from $(0,0)$ to $(n,m)$ avoiding points $(x,y)$ with $y > m+1-n+x$.

The number of these paths equals the total number of up/right paths from $(0,0)$ to $(n,m)$ minus the number of up/right paths from $(0,0)$ to $(n,m)$ that *do* pass through at least one point $(x,y)$ with $y > m+1-n+x$. The first of these numbers is $\binom{m+n}{m}$, as before. It remains to calculate the second of these numbers.

To that end, we first note that the reflection of $(n,m)$ across the line $y = m+2-n+x$ is $(n-2, m+2)$. Now there is a one-to-one correspondence between up/right paths from $(0,0)$ to $(m,n)$ that pass through something with $y \geq m+2-n+x$ with up/right paths from $(0,0)$ to $(n-2,m+2)$: indeed, taking a path of one kind, we may isolate the first point on it that lies on the line $y = m+2-n+x$, and reflect the rest of the path through that line, to obtain a path of the other kind. There are therefore $\binom{m+n}{m+2}$ paths of either kind.

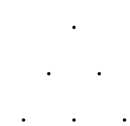
The answer is therefore

$$\binom{m+n}{m} - \binom{m+n}{m+2}$$

or, if you happen to prefer,

$$\frac{(m+n)!\,[(m+2)(m+1) - n(n-1)]}{(m+2)!\,n!}.$$

4. **[15]** In this problem and the next, the lawn consists of points in a triangular grid of size $n$, so that for $n = 3$ the lawn looks like

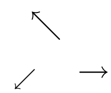


Mobots are allowed to be oriented to the east, 30° west of north, or 30° west of south. Under these conditions, for any given $n$, what is the minimum number of mobots needed to now the lawn?

**Answer:** $n$

**Solution:** It is evident that $n$ mobots are enough; just place one at the left edge of each row, and set them to move to the right. Suppose for the sake of contradiction that, for some $n$, it is possible to mow the lawn with fewer than $n$ mobots. Consider the minimum $n$ for which this is the case. Clearly $n > 1$. Now consider the mobot that mows the southwest corner of the lawn. This mobot is confined to either the bottom row or the left edge of the lawn; let's assume it's the former (the latter case is very similar). This mobot starts in the bottom row. Let's throw it away. Now if any remaining mobots start in the bottom row, they must not be set to move east, and we may advance them by one step in the direction they're set to go in, so that they land either off the lawn or in the second-to-last row. In doing so we have produced a starting formation of fewer than $n-1$ mobots in the first $n-1$ rows that will mow those rows. This contradicts the assumed minimality of $n$.

5. **[25]** With the same lawn and the same allowable mobot orientations as in the previous problem, let us call a formation "happy" if it is invariant under 120° rotations. (A rotation applies both to the positions of the mobots and to their orientations.) An example of a happy formation for $n = 2$ might be



Find the number of happy formations for a given $n$.

**Solution:** If $n \equiv 1 \pmod 3$, then there is a clump of grass at the center of the lawn; otherwise there are 3 blades of grass equally closest to the center. In the former case, whatever mobot mows this center blade of grass cannot possibly have a counterpart under a 120° rotation: if this mobot starts in the center, it must be oriented a certain way, and whatever way that is, cannot remain invariant under such a rotation; but if the mobot does not start in the center, then it will collide in the center with its counterparts under the 120° rotations.

In the case that $n \not\equiv 1 \pmod 3$, by considering how the three central blades of grass can be mowed, we easily see that there are two possibilities, both of which involve starting one mobot at each of these three positions. Both possibilities distinctly have the effect of cutting up the rest of the lawn (i.e., the part of the lawn not mowed by any of these three mobots) into three congruent pieces. The point is that these pieces are equivalent to trapezoids as in problem 3, both in their shape, and in the allowed directions of mobot motion. Luckily, we know how to count the ways to configure each such trapezoid. Call the trapezoid defined in problem 3 an "$(m, n)$-trapezoid."

If $n \equiv 0 \pmod 3$, then these pieces are either all $(\frac{2n}{3}, \frac{n-3}{3})$-trapezoids or all $(\frac{2n-3}{3}, \frac{n}{3})$-trapezoids, depending on how the central mobots are configured. In this case, the answer is

$$\left(\binom{n-1}{\frac{2n}{3}} - \binom{n-1}{\frac{2n+6}{3}}\right)^3 + \left(\binom{n-1}{\frac{2n-3}{3}} - \binom{n-1}{\frac{2n+3}{3}}\right)^3.$$

If, on the other hand, $n \equiv 2 \pmod 3$, then these pieces are either all $(\frac{2n-1}{3}, \frac{n-2}{3})$-trapezoids or all $(\frac{2n-4}{3}, \frac{n+1}{3})$-trapezoids, again depending on how the central mobots are configured. In this case, the answer is

$$\left(\binom{n-1}{\frac{2n-1}{3}} - \binom{n-1}{\frac{2n+1}{3}}\right)^3 + \left(\binom{n-1}{\frac{2n-4}{3}} - \binom{n-1}{\frac{2n-2}{3}}\right)^3.$$

## Polygons [110]

6. **[15]** Let $n$ be an integer at least 5. At most how many diagonals of a regular $n$-gon can be simultaneously drawn so that no two are parallel? Prove your answer.

**Answer:** $n$

**Solution:** Let $O$ be the center of the $n$-gon. Let us consider two cases, based on the parity of $n$:

- *$n$ is odd.* In this case, for each diagonal $d$, there is exactly one vertex $D$ of the $n$-gon, such that $d$ is perpendicular to line $OD$; and of course, for each vertex $D$, there is at least one diagonal $d$ perpendicular to $OD$, because $n \geq 5$. The problem of picking a bunch of $d$'s so that no two are parallel is thus transmuted into one of picking a bunch of $d$'s so that none of the corresponding $D$'s are the same. Well, go figure.
- *$n$ is even.* What can I say? For each diagonal $d$, the perpendicular dropped from $O$ to $d$ either passes through two opposite vertices of the $n$-gon, or else bisects two opposite sides. Conversely, for each line joining opposite vertices or bisecting opposite sides, there is at least one diagonal perpendicular to it, because $n \geq 6$. By reasoning similar to the odd case, we find the answer to be $n$.

7. **[25]** Given a convex $n$-gon, $n \geq 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal either in the interior of the $n$-gon or at a vertex? Prove your answer.

**Answer:** If $n = 4$, then 2; otherwise, $n$.
**Solution:** First of all, assume without loss of generality that the $n$-gon is regular (this has no effect as far as diagonal intersection is concerned). Also, treat $n = 4$ as a special case; obviously the answer is 2 here.

If $n$ is odd, simply draw $n$ diagonals, connecting each vertex to the ones $(n-1)/2$ vertices away (in either direction).

If $n$ is even, first draw the $n/2$ diagonals connecting pairs of vertices $n/2$ vertices apart. Then, there are $n$ diagonals connecting pairs of vertices $n/2 - 1$ vertices apart; they come in $n/2$ pairs of parallel diagonals; from each such pair, randomly pick one diagonal and draw it.

To see that these constructions work, note that two diagonals, each connecting pairs of vertices at least $n/2 - 1$ vertices apart, can fail to intersect or share a vertex only if they are parallel.

The previous problem shows that these constructions are optimal.

8. **[15]** Given a regular $n$-gon with sides of length 1, what is the smallest radius $r$ such that there is a non-empty intersection of $n$ circles of radius $r$ centered at the vertices of the $n$-gon? Give $r$ as a formula in terms of $n$. Be sure to prove your answer.

**Answer:** $r = \frac{1}{2} \csc \frac{180°}{n}$
**Solution:** It is easy to see that, with this $r$, all the circles pass through the center of the $n$-gon. The following proves that this $r$ is necessary even if the word "circle" is replaced by the word "disk."

For $n$ even, it is easy to see using symmetry that containing the center point is necessary and sufficient. For $n$ odd, there is more work to do. Again, containing the center point is sufficient. To see its necessity, consider three circles: a circle at a vertex $A$, and the two circles on the segment $BC$ opposite $A$. Circles $B$ and $C$ intersect in a region $R$ symmetric about the perpendicular bisector of $BC$, with the closest point of $R$ to $A$ being on this line. Hence, circle $A$ must intersect $R$ at some point on the perpendicular bisector of $BC$; and thus we see the entire perpendicular bisector of $BC$

inside of the $n$-gon is contained in the circles. Now, this bisector contains the center of the $n$-gon, so some circle must contain the center. But by symmetry, if one circle contains the center, all do. Thus, in any case, it is necessary and sufficient for $r$ to be large enough so the center is contained in a circle. Basic trigonometry gives the answer, which equals the distance between a vertex and the center.

9. [**40**] Let $n \geq 3$ be a positive integer. Prove that given any $n$ angles $0 < \theta_1, \theta_2, \ldots, \theta_n < 180°$, such that their sum is $180(n-2)$ degrees, there exists a convex $n$-gon having exactly those angles, in that order.

   **Solution:** We induct on $n$. The statement holds trivially for $n = 3$, as all triangles are convex. Now, suppose that the statement is true for $n-1$, where $n \geq 4$. Let $\theta_1, \theta_2, \ldots, \theta_n$ be $n$ angles less than $180°$ whose sum equals $180(n-2)$ degrees. The statement is clearly true if $n = 4$ and $\theta_1 = \theta_3 = 180° - \theta_2 = 180° - \theta_4$ since we can easily form a parallelogram, so assume otherwise.

   I claim that there exist two adjacent angles whose sum is greater than $180°$. Assume otherwise. Then, we have $\theta_i + \theta_{i+1} \leq 180$ for $i = 1, 2, \ldots, n$, where $\theta_{n+1} = \theta_1$. Summing these inequalities over all $i$ yields $2 \cdot 180(n-2) \leq 180n$, which is equivalent to $n \leq 4$. Of course, we can have $n = 4$ if and only if we have equality in each of the above inequalities, forcing us to have a parallelogram contrary to our assumption.

   Hence, we have two adjacent angles with sum greater than $180°$. Without loss of generality, let these angles be $\theta_{n-1}$ and $\theta_n$, relabeling if necessary. By the inductive hypothesis, we may construct an $(n-1)$-gon with angles $\theta_1, \theta_2, \ldots, \theta_{n-2}, \theta_{n-1} + \theta_n - 180°$, as these angles are each less than $180°$ and their sum equals $180(n-3)$ degrees. Consider the vertex with angle $\theta_{n-1} + \theta_n - 180°$. Note that we can "clip off" a triangle with angles $\theta_{n-1} + \theta_n - 180°$, $180° - \theta_{n-1}$, and $180° - \theta_n$ at this vertex, yielding an $n$-gon with the desired angles, completing the inductive step.

10. [**15**] Suppose we have an $n$-gon such that each interior angle, measured in degrees, is a positive integer. Suppose further that all angles are less than $180°$, and that all angles are different sizes. What is the maximum possible value of $n$? Prove your answer.

    **Answer:** 26
    **Solution:** Let's work with the exterior angles (each is 180 minus the interior angle). Then the conditions on the exterior angles are identical to the conditions on the interior angles: each is a positive integer between 1 and 179 inclusive. The sum of the exterior angles is exactly 360. However, the sum of 1 through 27 is $27 \cdot 28/2 = 378$, which is too large. We can get 26 using angles of 1 through 25 (sum 325) and an angle of 35. The previous problem shows that this is actually possible.

## What do the following problems have in common? [170]

11. [**15**] The lottery cards of a certain lottery contain all nine-digit numbers that can be formed with the digits 1, 2 and 3. There is exactly one number on each lottery card. There are only red, yellow and blue lottery cards. Two lottery numbers that differ from each other in all nine digits always appear on cards of different color. Someone

draws a red card and a yellow card. The red card has the number 122 222 222 and the yellow card has the number 222 222 222. The first prize goes to the lottery card with the number 123 123 123. What color(s) can it possibly have? Prove your answer.

**Answer:** The card with the number 123 123 123 is red.
**Solution:** First, it can in fact be red, if, say, cards are colored based on the first digit only (1 = red, 2 = yellow, 3 = blue). We now endeavor to show it must be red.

Consider the cards 333 133 133 and 331 331 331: they each differ in all their digits from 122 222 222 and from 222 222 222, so they must both be blue. Now 211 311 311 differs in all its digits from both 122 222 222 and 333 133 133, so it must be yellow. Finally, 123 123 123 differs in all its digits from both 331 331 331 and 211 311 311, so it must be red.

12. [**25**] A $3 \times 3 \times 3$ cube is built from 27 unit cubes. Suddenly five of those cubes mysteriously teleport away. What is the minimum possible surface area of the remaining solid? Prove your answer.

**Answer:** 50
**Solution:** Orient the cube so that its edges are parallel to the $x$-, $y$-, and $z$-axes. A set of three unit cubes whose centers differ only in their $x$-coordinate will be termed an "$x$-row"; there are thus nine $x$-rows. Define "$y$-row" and "$z$-row" similarly.

To achieve 50, simply take away one $x$-row and one $y$-row (their union consists of precisely five unit cubes).

To show that 50 is the minimum: Note that there cannot be two $x$-rows that are both completely removed, as that would imply removing six unit cubes. (Similar statements apply for $y$- and $z$-rows, of course.) It is also impossible for there to be one $x$-row, one $y$-row, and one $z$-row that are all removed, as that would imply removing seven unit cubes. Every $x$-, $y$-, or $z$-row that is not completely removed contributes at least 2 square units to the surface area. Thus, the total surface area is at least $9 \cdot 2 + 8 \cdot 2 + 8 \cdot 2 = 50$.

13. [**40**] Having lost a game of checkers and my temper, I dash all the pieces to the ground but one. This last checker, which is perfectly circular in shape, remains completely on the board, and happens to cover equal areas of red and black squares. Prove that the center of this piece must lie on a boundary between two squares (or at a junction of four).

**Solution:** Suppose, for the sake of contradiction, that the problem is false. Evidently, at least one boundary between adjacent squares must lie within our checker, or else the checker would exist entirely within one square, meaning it would cover only one color. Note also that a checker's diameter is smaller than the side of any square of the board, so there are at most two such boundaries within our checker (one in each direction). Let $\ell$ be this, or one of these, boundaries. Draw a diameter $d$ of the checker parallel to $\ell$. Presumably, the strip of the checker between $\ell$ and $d$ is part red, part black. These red and black areas are unequal, however, because the center of the checker does not lie on *any* boundary between squares. But, if we were to swap colors within this strip, *then* the checker *would* have equal red and black areas, because then it would be colored in a way such that flipping it across $d$ swaps the colors. This shows that, the way it is *currently* colored, the checker does *not* have equal red and black areas. This gives us the desired contradiction.

14. [**40**] A number $n$ is called *bummed out* if there is exactly one ordered pair of positive integers $(x, y)$ such that

$$\lfloor x^2/y \rfloor + \lfloor y^2/x \rfloor = n.$$

Find all bummed out numbers.

**Answer:** 2, 6, 8, 10
**Solution:** Suppose $n$ is bummed out. If $(a, b)$ is one solution for $(x, y)$ to the given equation $\lfloor x^2/y \rfloor + \lfloor y^2/x \rfloor = n$, then $(b, a)$ is another, so the unique solution $(a, b)$ better have the property that $a = b$ and $n = 2a \geq 2$. In particular, $n$ is an even positive integer.

Now, if $n = 2a \geq 12$, then setting $x = a - 1 \geq 5$, $y = a + 1 \geq 7$, we have

$$\left\lfloor \frac{x^2}{y} \right\rfloor + \left\lfloor \frac{y^2}{x} \right\rfloor = \left\lfloor a - 3 + \frac{4}{a+1} \right\rfloor + \left\lfloor a + 3 + \frac{4}{a-1} \right\rfloor = 2a = n,$$

so $n$ cannot be bummed out.

Moreover, $\lfloor 1^2/2 \rfloor + \lfloor 2^2/1 \rfloor = 4$, so 4 is not bummed out. The only possibilities left are 2, 6, 8, and 10.

To check these, note that

$$n = \left\lfloor \frac{x^2}{y} \right\rfloor + \left\lfloor \frac{y^2}{x} \right\rfloor > -2 + \frac{x^2}{2y} + \frac{x^2}{2y} + \frac{y^2}{x} \geq -2 + \frac{3x}{\sqrt[3]{4}}$$

so

$$x < \frac{\sqrt[3]{4}}{3}(n+2) < .53(n+2),$$

and similarly for $y$. So we only have to check $x, y \leq \lfloor .53(10 + 2) \rfloor = 6$:

| $x \setminus y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 9 | 16 | 25 | 36 |
| 2 | 4 | 4 | 5 | 9 | 12 | 18 |
| 3 | 9 | 5 | 6 | 7 | 9 | 13 |
| 4 | 16 | 9 | 7 | 8 | 9 | 11 |
| 5 | 25 | 12 | 9 | 9 | 10 | 11 |
| 6 | 36 | 18 | 13 | 11 | 11 | 12 |

15. [**50**] Find, with proof, all positive integer palindromes whose square is also a palindrome.

**Answer:** A palindrome satisfies the requirement if and only if the sum of the squares of its digits is less than 10. We may categorize these numbers this way:

- 3
- Any palindromic combination of 1s and 0s with at most nine 1s.

- Any palindrome consisting of a single 2 in the middle and 1s and 0s elsewhere, with at most four 1s.
- $2000\ldots0002$
- $2000\ldots0001000\ldots0002$

**Solution:** Let $n := \sum_{i=0}^{d} a_i \cdot 10^i$ be a palindrome, where the $a_i$ are digits with $a_i = a_{d-i}$ and $a_d \neq 0$. Then, if we let

$$b_k := \sum_{i+j=k} a_i a_j$$

for all $0 \leq k \leq 2d$, then

$$n^2 = \sum_{k=0}^{2d} b_k \cdot 10^k$$

(this is not necessarily the decimal expansion of $n^2$, however). We have to show that $\sum_{i=0}^{d} a_i^2 < 10$ if and only if $n^2$ is a palindrome.

Suppose $\sum_{i=0}^{d} a_i^2 < 10$. Then, by the AM-GM inequality, we have

$$b_k = \sum_{i+j=k} a_i a_j \leq \sum_{i+j=k} \frac{a_i^2 + a_j^2}{2} \leq \sum_{i=0}^{d} \frac{a_i^2}{2} + \sum_{j=0}^{d} \frac{a_j^2}{2} < \frac{10}{2} + \frac{10}{2} = 10.$$

Thus, loosely speaking, no carrying is ever done in computing $n \times n$ by long multiplication, so the digit in the $10^k$ place in $n^2$ is precisely $b_k$, and it's easy to see that $b_k = b_{2d-k}$ and that $b_{2d} = a_d^2 \neq 0$. So $n^2$ is indeed a palindrome, as desired.

Now suppose $\sum_{i=0}^{d} a_i^2 \geq 10$. Here note that

$$b_d = \sum_{i+j=d} a_i a_j = \sum_{i=0}^{d} a_i a_{d-i} = \sum_{i=0}^{d} a_i^2 \geq 10.$$

Thus, it *cannot* be true that, for all $k$, $b_k$ represents the $10^k$ digit of $n^2$, because no digit can be greater than or equal to 10. Let $\ell$ be the greatest such that $b_\ell$ does not represent the $10^\ell$ digit of $n^2$. We are trying to prove that $n^2$ cannot be a palindrome. Consider three cases:

- $a_d = a_0 \geq 4$. In this case we must have $\ell \geq 2d$, because $b_{2d} = a_d^2 > 10$.
  If $a_0 = 4$, then $n^2$ ends in the digit 6, but lies in the interval $[16 \cdot 10^{2d}, 25 \cdot 10^{2d})$, and so starts with either a 1 or a 2; thus, $n^2$ cannot be a palindrome. Similarly, if $a_0 = 5$, then $n^2$ ends in 5 but starts with 2 or 3; if $a_0 = 6$, then $n^2$ ends in 6 but starts with 3 or 4; if $a_0 = 7$, then $n^2$ ends in 9 but starts with 4, 5, or 6; if $a_0 = 8$, then $n^2$ ends in 4 but starts with 6, 7 or 8; if $a_0 = 9$, then $n^2$ ends in 1 but starts with 8 or 9.
- $\ell \geq 2d$ and $a_d = a_0 \leq 3$.
  Here we do something similar, but with a slight twist. The units digit of $n^2$ is $a_0^2$. Because $\ell \geq 2d$, $n^2$ must be in the interval $[(a_0^2 + 1) \cdot 10^{2d}, (a_0 + 1)^2 \cdot 10^{2d})$,

which is certainly a subset of the interval $[(a_0^2 + 1) \cdot 10^{2d}, a_0^2 \cdot 10^{2d+1})$. No integer in even this larger interval manages to start with the digit $a_0^2$, so $n^2$ cannot be palindromic.

- $\ell < 2d$.

  Here we can rest assured that $n^2$ does have $(2d + 1)$ digits — that is, the first digit is in the $10^{2d}$ place. In order for $n^2$ to be a palindrome, the digits in the $10^k$ and $10^{2d-k}$ places must always be the same.

  Now $b_\ell, b_{\ell+1}, \ldots, b_{2d}$ had all better be less than 10, or else $\ell$ would be greater than what it is. Thus, the numbers just listed *do* appear as the *lowest* digits of $n^2$ in left-to-right order, although they *don't* appear as the *highest* $(2d + 1 - \ell)$ digits of $n^2$ in right-to-left order. Thus, $n^2$ cannot be a palindrome.