IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Team Round B: Solutions

Mobotics [135]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged in an $m \times n$ rectangular grid, that is, with m rows running east-west and n columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x < n \text{ and } 0 \leq y < m\}.$$

Our machinery consists of a fleet of identical *mowbots* (or "mobots" for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction, either north or east (*not* south or west). At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered "different" if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot starts on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

As an example, one allowable formation for $m = 2$, $n = 3$ might be as follows:

$$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \uparrow & \rightarrow & \cdot \end{array}$$

1. **[25]** Prove that the maximum number of mobots you need to mow your lawn is $m+n-1$.

   Solution: This is attainable if we place one mobot at each clump in the first row, oriented north, and one mobot in the first column of each row except the first, oriented east.

   To show that at most $m + n - 1$ mobots can be used, note that each mobot must mow at least one of the $m + n - 1$ clumps in union of the first row and last column.

2. **[40]** Prove that the minimum number of mobots you need to mow your lawn is $\min\{m, n\}$.

   Solution: This is attainable if we place one mobot at each clump in the first column, oriented toward the east, or one mobot at each clump in the last row, oriented toward the north (whichever of the two is more efficient).

   To show that at least $\min\{m, n\}$ mobots must be used, note that each of the clumps $(0, 0)$, $(1, 1)$, $(2, 2)$, $\ldots$, $(\min\{m, n\} - 1, \min\{m, n\} - 1)$ must be mowed by a different mobot.

3. **[15]** Prove that, given any formation, each mobot may be colored in one of three colors — say, white, black, and blue — such that no two adjacent clumps of grass are mowed by different mobots of the same color. Two clumps of grass are adjacent if the distance between them is 1. In your proof, you may use the Four-Color Theorem if you're familiar with it.

   Solution: We can divide the coordinate plane into regions: Let's say a point belongs to Region 0 if the closest lattice point to it is not on the lawn, and each mobot M owns a region that is the set of points for which the closest lattice point is on the lawn and mowed by M. Applying the Four-Color Theorem to these regions, we note that all the conditions demanded in the problem are satisfied. In particular at most 3 colors are used on the mobots because every mobot region borders Region 0 and hence is not colored the same color as Region 0.

4. **[15]** For $n = m = 4$, find a formation with 6 mobots for which there are exactly 12 ways to color the mobots in three colors as in problem 3. (No proof is necessary.)

   Solution: Place north-oriented mobots at $(0, 0)$, $(1, 0)$, $(2, 2)$, and $(3, 2)$, and east-oriented mobots at $(2, 0)$ and $(2, 1)$. Other formations are possible.

5. **[40]** For $n, m \geq 3$, prove that a formation has exactly six possible colorings satisfying the conditions in problem 3 if and only if there is a mobot that starts at $(1, 1)$.

   Solution: Let's trace a path through all the mobots. There must be a mobot at $(0, 0)$, so start the path there. If that mobot is oriented to move up, then move to the right; if that mobot is oriented to move to the right, then move up. In doing so, you will meet another mobot, upon which you can repeat the above process, until you leave the lawn.

   In tracing this path, we've effectively ordered the mobots, starting with $(0, 0)$. At any point in this ordering, we have a number of choices for how to color the current mobot without violating the condition with any previously colored mobot. Specifically, there are three ways to color the mobot at $(0, 0)$. If a mobot has x- or y-coordinate 0, there are 2 ways to color it: it need only not be the same color as the previous mobot on our trail. If a mobot has both x- and y-coordinates positive, then there is only 1 way to color it: it cannot be the same color as whatever mobots mows the clump directly south or directly west of its starting point.

   Thus, if there is a mobot at $(1, 1)$, our path must begin with either $(0, 0)$–$(0, 1)$–$(1, 1)$ or $(0, 0)$–$(1, 0)$–$(1, 1)$, and in either case there are only 6 ways to do the coloring. Otherwise, if there is no mobot at $(1, 1)$, our path must begin with either $(0, 0)$–$(0, 1)$–$(0, 2)$ or $(0, 0)$–$(1, 0)$–$(2, 0)$, and in either case there are already $3 \times 2 \times 2 = 12$ ways to do the coloring thus far.

Polygons [130]

6. **[15]** Suppose we have a regular hexagon and draw all its sides and diagonals. Into how many regions do the segments divide the hexagon? (No proof is necessary.)

Answer: 24
Solution: An accurate diagram and a careful count yields the answer.

7. **[25]** Suppose we have an octagon with all angles of 135°, and consecutive sides of alternating length 1 and $\sqrt{2}$. We draw all its sides and diagonals. Into how many regions do the segments divide the octagon? (No proof is necessary.)

   Answer: 84
   Solution: The easiest way to see the answer is to view the octagon as five unit squares in a cross arrangement, with four half-squares wedged at the corners. The center square is divided into 8 regions. The other 4 squares are each divided into 15 regions. The 4 half-squares are each divided into 4 regions. The answer is thus $8 + 4 \times 15 + 4 \times 4 = 84$.

8. **[25]** A regular 12-sided polygon is inscribed in a circle of radius 1. How many chords of the circle that join two of the vertices of the 12-gon have lengths whose squares are rational? (No proof is necessary.)

   Answer: 42
   Solution: The chords joining vertices subtend minor arcs of 30°, 60°, 90°, 120°, 150°, or 180°. There are 12 chords of each of the first five kinds and 6 diameters. For a chord with central angle θ, we can draw radii from the two endpoints of the chord to the center of the circle. By the law of cosines, the square of the length of the chord is $1 + 1 - 2\cos\theta$, which is rational when θ is 60°, 90°, 120°, or 180°. The answer is thus $12 + 12 + 12 + 6 = 42$.

9. **[25]** Show a way to construct an equiangular hexagon with side lengths 1, 2, 3, 4, 5, and 6 (not necessarily in that order).

   Solution: The trick is to view an equiangular hexagon as an equilateral triangle with its corners cut off. Consider an equilateral triangle with side length 9, and cut off equilateral triangles of side length 1, 2, and 3 from its corners. This yields an equiangular hexagon with sides of length $1, 6, 2, 4, 3, 5$ in that order.

10. **[40]** Given a convex n-gon, $n \geq 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal strictly in the interior of the n-gon? Prove that your answer is correct.

    Answer: $\lfloor n/2 \rfloor$
    Solution: If n is even, simply draw all $n/2$ diagonals connecting a vertex to the one $n/2$ vertices away. If n is odd, pretend one of the vertices does not exist, and do the above for the $(n-1)$-gon remaining.

    To show this is optimal, consider any given drawn diagonal: it divides the remaining $n-2$ vertices into two camps, one of which therefore has size at most $\lfloor n/2 \rfloor - 1$, and one cannot draw two diagonals sharing a vertex.

What do the following problems have in common? [135]

11. **[15]** Find the largest positive integer n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square. Prove that your answer is correct.

Answer: 3
Solution: Clearly $1! + 2! + 3! = 9$ works. For $n \geq 4$, we have

$$1! + 2! + 3! + \cdots + n! \equiv 1! + 2! + 3! + 4! \equiv 3 \pmod 5,$$

but there are no squares congruent to 3 modulo 5.

12. [**15**] Find all ordered triples (x, y, z) of positive reals such that $x + y + z = 27$ and $x^2 + y^2 + z^2 - xy - yz - zx = 0$. Prove that your answer is correct.

Answer: $(9, 9, 9)$
Solution: We have $x^2 + y^2 + z^2 - xy - yz - zx = \dfrac{(x-y)^2 + (y-z)^2 + (z-x)^2}{2} = 0$, which implies $x = y = z = \dfrac{27}{3} = 9$.

13. [**25**] Four circles with radii $1, 2, 3$, and r are externally tangent to one another. Compute r. (No proof is necessary.)

Answer: $6/23$
Solution: Let A, B, C, P be the centers of the circles with radii 1, 2, 3, and r, respectively. Then, ABC is a 3-4-5 right triangle. Using the law of cosines in $\triangle PAB$ yields

$$\cos \angle PAB = \frac{3^2 + (1+r)^2 - (2+r)^2}{2 \cdot 3 \cdot (1+r)} = \frac{3-r}{3(1+r)}$$

Similarly,

$$\cos \angle PAC = \frac{4^2 + (1+r)^2 - (3+r)^2}{2 \cdot 4 \cdot (1+r)} = \frac{2-r}{2(1+r)}$$

We can now use the equation $(\cos \angle PAB)^2 + (\cos \angle PAC)^2 = 1$, which yields $0 = 23r^2 + 132r - 36 = (23r - 6)(r + 6)$, or $r = 6/23$.

14. [**40**] Find the prime factorization of

$$2006^2 \cdot 2262 - 669^2 \cdot 3599 + 1593^2 \cdot 1337.$$

(No proof is necessary.)

Answer: $2 \cdot 3 \cdot 7 \cdot 13 \cdot 29 \cdot 59 \cdot 61 \cdot 191$
Solution: Upon observing that $2262 = 669 + 1593$, $3599 = 1593 + 2006$, and $1337 = 2006 - 669$, we are inspired to write $a = 2006, b = 669, c = -1593$. The expression in question then rewrites as $a^2(b-c) + b^2(c-a) + c^2(a-b)$. But, by experimenting in the general case (e.g. setting $a = b$), we find that this polynomial is zero when two of a, b, c are equal. Immediately we see that it factors as $(b-a)(c-b)(a-c)$, so the original expression is a way of writing $(-1337) \cdot (-2262) \cdot (3599)$. Now, $1337 = 7 \cdot 191$, $2262 = 2 \cdot 3 \cdot 13 \cdot 29$, and $3599 = 60^2 - 1^2 = 59 \cdot 61$.

15. [**40**] Let a, b, c, d be real numbers so that c, d are not both 0. Define the function

$$m(x) = \frac{ax + b}{cx + d}$$

on all real numbers x except possibly $-d/c$, in the event that $c \neq 0$. Suppose that the equation $x = m(m(x))$ has at least one solution that is not a solution of $x = m(x)$. Find all possible values of $a + d$. Prove that your answer is correct.

Answer: 0
Solution: That 0 is a possible value of $a + d$ can be seen by taking $m(x) = -x$, i.e., $a = -d = 1$, $b = c = 0$. We will now show that 0 is the only possible value of $a + d$.

The equation $x = m(m(x))$ implies $x = \dfrac{(a^2 + bc)x + (a + d)b}{(a + d)cx + (bc + d^2)}$, which in turn implies

$$(a + d)[cx^2 + (-a + d)x - b] = 0.$$

Suppose for the sake of contradiction that $a + d \neq 0$. Then the above equation would further imply

$$cx^2 + (-a + d)x - b = 0, x(cx + d) = ax + b,$$

which would imply $x = m(x)$ for any x except possibly $-d/c$. But of course $-d/c$ is not a root of $x = m(m(x))$ anyway, so in this case, all solutions of $x = m(m(x))$ are also solutions of $x = m(x)$, a contradiction. So our assumption was wrong, and in fact $a + d = 0$, as claimed.