Recall: Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Deny (cooperate)</th>
<th>Confess (betray)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deny (cooperate)</td>
<td>-1, -1</td>
<td>-3, 0</td>
</tr>
<tr>
<td>Confess (betray)</td>
<td>0, -3</td>
<td>-2, -2</td>
</tr>
</tbody>
</table>

("-1" = “1 year in jail")

Our prediction: both prisoners will **confess**

**Why?**
No matter what the other player may play, confessing is optimal for me.

Playing confess is a **dominant strategy equilibrium**
Recall: Rock-Paper-Scissors

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>0,0</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>Paper</td>
<td>1,-1</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>Scissors</td>
<td>-1,1</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Our prediction: both players play uniformly at random

Why?

*If my opponent plays uniformly at random, then playing uniformly at random is optimal for me.*

Player u.a.r. is a **Nash equilibrium**

Remarks:
1. Nash is a much *weaker* solution of a game compared to dominant strategy equilibrium
   - need assumption/knowledge about other player’s strategy to justify my strategy
2. No dominant strategy equilibrium exists in Rock-Paper-Scissors
3. No Nash equilibrium exists in pure (i.e. non-randomizing) strategies
4. There is a unique Nash equilibrium in this game
## Football vs Theater

<table>
<thead>
<tr>
<th></th>
<th>5/6</th>
<th>1/6</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Insist on Theater</td>
<td>Accept Football</td>
</tr>
<tr>
<td>Accept Theater</td>
<td>1, 5</td>
<td>0, 0</td>
</tr>
<tr>
<td>Insist on Football</td>
<td>0, 0</td>
<td>5, 1</td>
</tr>
</tbody>
</table>

Our prediction here?
- there are two obvious Nash equilibria
- there is a 3rd Nash equilibrium $x_{\text{blue}} = \left( \frac{1}{6}, \frac{5}{6} \right)$, $x_{\text{orange}} = \left( \frac{5}{6}, \frac{1}{6} \right)$

**cool fact:** in two-player (non-degenerate games) there is always an odd number of Nash eq
Our focus (part I): Normal-Form Games

Normal-form Games: Single-shot, simultaneous move, complete information Games

Complete-information means:

• Every player knows their own objective as well as the objective of every other player

[If I throw away structure and represent this game as a huge table, whose rows/columns are all possible algorithms (a.k.a. contingency plans) that the two players can use]
More Abstract Game Formulation

- **Def:** A *finite n-player game* is described by:
  - a set of pure strategies/actions per player: $S_p$
  - a utility/payoff function per player: $u_p : x_q S_q \rightarrow \mathbb{R}$

- **Def:** A *randomized/mixed strategy* for player $p$ is any $x_p \in \Delta^{S_p}$
  - assigns probability $x_p(j)$ to each $j \in S_p$
  - i.e. $\Delta^{S_p}$ is the simplex whose vertices are identified with the elements of $S_p$

- **Def:** a player’s *expected utility* is
  - $u_p(x_1, \ldots, x_n) = \sum_{s \in x_q S_q} u_p(s) x_1(s_1) \cdot \ldots \cdot x_n(s_n)$

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<thead>
<tr>
<th></th>
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<th>1/6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theater fine</td>
<td>1, 5</td>
<td>0, 0</td>
</tr>
<tr>
<td>Football</td>
<td>0, 0</td>
<td>5, 1</td>
</tr>
</tbody>
</table>

$x_{\text{blue}} = \left(\frac{1}{6}, \frac{5}{6}\right)$   $x_{\text{orange}} = \left(\frac{5}{6}, \frac{1}{6}\right)$

$u_{\text{blue}} = \frac{1}{6} \cdot \frac{5}{6} \cdot 1 + \frac{5}{6} \cdot \frac{1}{6} \cdot 5 = \frac{5}{6}$

$u_{\text{orange}} = \frac{1}{6} \cdot \frac{5}{6} \cdot 5 + \frac{5}{6} \cdot \frac{1}{6} \cdot 1 = \frac{5}{6}$
More Abstract Game Formulation

• **Def:** A *finite n-player game* is described by:
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  - $u_p(x_1, ..., x_n) = \sum_{s \in \times_q S_q} u_p(s) x_1(s_1) \cdot \ldots \cdot x_n(s_n)$

A piece of very useful notation: if $x_1, ..., x_n$ are player strategies, then $x_{-i}$ denotes the strategies of all players except player $i$’s

• **Def:** a collection of mixed strategies $x_1, ..., x_n$ is a *Nash equilibrium* iff
  - $\forall i, x_i' : u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i})$

• **Def:** a collection $x_1, ..., x_n$ is a *dominant strategy equilibrium* iff
  - $\forall i, x_i', x_{-i}' : u_i(x_i, x_{-i}') \geq u_i(x_i', x_{-i})$
More Abstract Game Formulation

- **Def:** a collection $x_1, \ldots, x_n$ is a *Nash equilibrium* iff
  - $\forall i, x_i': u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i})$

- **Def:** a collection $x_1, \ldots, x_n$ is a *dominant strategy equilibrium* iff
  - $\forall i, x'_i, x'_{-i}: u_i(x_i, x'_{-i}) \geq u_i(x'_i, x'_{-i})$

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$x_{blue} = \left( \frac{1}{6}, \frac{5}{6} \right)$  
$x_{orange} = \left( \frac{5}{6}, \frac{1}{6} \right)$

\[
\begin{align*}
 u_{blue}(x_{blue}, x_{orange}) &= \frac{1}{6} \cdot \frac{5}{6} \cdot 1 + \frac{5}{6} \cdot \frac{1}{6} \cdot 5 = \frac{5}{6} \\
 u_{blue}(\text{theater fine}', x_{orange}) &= \frac{5}{6} \cdot 1 + \frac{1}{6} \cdot 0 = \frac{5}{6} \\
 u_{blue}(\text{football!}', x_{orange}) &= \frac{5}{6} \cdot 0 + \frac{1}{6} \cdot 5 = \frac{5}{6}
\end{align*}
\]

\[u_{blue}(x_{blue}', x_{orange}) = \frac{5}{6}, \forall x_{blue}'\]
Nash’s Theorem

[Nash 1950]: Every finite game (i.e. with a finite number of players and a finite number of pure strategies per player) has a Nash equilibrium.

• We’ll prove it!
• We’ll make use of Brouwer’s fixed point theorem, following a proof that Nash produced in 1951; his original proof used Kakutani’s fixed point theorem.
Menu
• Refresher and game-theoretic formalism
• Nash’s theorem
• von Neumann’s theorem
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• von Neumann’s theorem
Brouwer’s Fixed Point Theorem

[Brouwer 1910]: Let $f : D \to D$ be a continuous function from a convex and compact subset $D$ of the Euclidean space to itself. Then there exists an $x \in D$ s.t. $x = f(x)$.

N.B. All conditions in the statement of the theorem are necessary.

Below we show a few examples, when $D$ is the 2-dimensional disk.
Brouwer’s Fixed Point Theorem
Brouwer’s Fixed Point Theorem
Brouwer’s Fixed Point Theorem
Brouwer ⇒ Nash
Visualizing Nash’s Proof

Penalty Shot Game

<table>
<thead>
<tr>
<th>Kick</th>
<th>Left</th>
<th>Right</th>
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<tbody>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>Left</td>
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<td>-1, 1</td>
</tr>
<tr>
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\( f: [0,1]^2 \rightarrow [0,1]^2 \), continuous such that fixed points \( \equiv \) Nash eq.
Visualizing Nash’s Proof

Penalty Shot Game

Kick Dive

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Visualizing Nash’s Proof

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Visualizing Nash’s Proof

Penalty Shot Game

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Visualizing Nash’s Proof

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<th>Right</th>
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<tbody>
<tr>
<td>Dive</td>
<td>½</td>
<td>½</td>
</tr>
<tr>
<td>Left</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
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<td>-1, 1</td>
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Penalty Shot Game

Real proof: on the board
Every finite game has a Nash Equilibrium.

Proof: 1. Define a function $s: \Delta^q \times \Delta^q \rightarrow \Delta^q \times \Delta^q$ as

$$s^q(i, y) = \left( \frac{x^q_i + \text{Gain}_{x^q_i}(y)}{1 + \sum_{s^q_i} \text{Gain}_{x^q_i}(y)} \right)$$

where $y_i$ is the best response of player $i$ to $x_i^q$.

Note: $x_i^q$ is valid if and only if

$$\sum_{s^q_i} \text{Gain}_{x^q_i}(y) > 0$$

2. $s$ is continuous in $\Delta$ (exercise to complete)

3. Conclusion: fixed point of $s$ is Nash Equilibrium.

Proof: Let $x^q$ be the fixed point

\[ x^q : \text{Nash Equilibrium} \]

Suppose this is not true.

Then there is some $i$ such that $x^q_i < 0$.

Thus, we can choose $x^q_i$ such that

$$x^q_i > 0$$

and

$$\sum_{s^q_i} \text{Gain}_{x^q_i}(y) > 0$$

in particular,

$$\text{Gain}_{x^q_i}(x^q_i) = 0$$

But

$$x^q_i(s^q_i) = \frac{x^q_i(s^q_i) + \text{Gain}_{x^q_i}(s^q_i)}{1 + \sum_{s^q_i} \text{Gain}_{x^q_i}(y)} < x^q_i(s^q_i)$$

Contradiction!!
Menu

• Refresher and game-theoretic formalism
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  • Brouwer’s theorem
  • Nash from Brouwer
• von Neumann’s theorem
Menu

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Two-player games

- **Def:** A *finite n-player game* is described by:
  - a set of pure strategies/actions per player: $S_p$
  - a utility/payoff function per player: $u_p: \times_q S_q \rightarrow \mathbb{R}$

- A 2-player can be summarized by two matrices $(R, C)_{m \times n}$
  - rows: indexed by pure strategies of “row player”
  - columns: indexed by pure strategies of “column player”

- Mixed strategy for row player: $x \in \Delta^m$
- Mixed strategy for column player: $y \in \Delta^n$
- Expected utility of row player: $u_{\text{row}}(x, y) = x^T R y = \sum_{ij} R_{ij} x_i y_j$
- Expected utility of column player: $u_{\text{column}}(x, y) = x^T C y = \sum_{ij} C_{ij} x_i y_j$
- $(x, y)$ is Nash equilibrium iff
  \[ \forall x': x^T R y \geq x'^T R y \]
  \[ \forall y': x^T C y \geq x^T C y' \]
Two-player Zero-Sum games

Minimax Theorem [von Neumann’28]: Consider a two-player game zero-sum game \((R, C)_{m \times n}\) i.e. 
\[ R + C = 0. \]
Then 
\[ \min_{x \in \Delta^m} \max_{y \in \Delta^n} x^T C y = \max_{y \in \Delta^n} \min_{x \in \Delta^m} x^T C y \quad (*) \]

Interpretation:

• (*) says: “If \(\forall y, \exists x\) s.t. \(x^T C y \leq v^* \Rightarrow \exists x, \forall y\) s.t. \(x^T C y \leq v^*\)”
• If \(x^*\) is argmin of LHS, \(y^*\) argmax of RHS, \(v^*\) optimal value of (*), then \((x^*, y^*)\) is a Nash equilibrium, i.e. if \(\min\) and \(\max\) adopt \(x^*\) and \(y^*\) then (i) \(\min\) pays \(v^*\) to \(\max\) and (ii) no player can improve by unilaterally deviating
• why? Because
  • under \((x^*, y^*)\) \(\min\) pays \(\max\) at most \(v^*\) (since \(v^*\) optimum of LHS and \(x^*\) is argmin)
  • under \((x^*, y^*)\) \(\max\) receives from \(\min\) at least \(v^*\) (since \(v^*\) optimum of RHS and \(y^*\) is argmax)
  • by the above two: under \((x^*, y^*)\) \(\min\) pays exactly \(v^*\) to \(\max\), hence (i) is proven
  • to prove (ii), suppose \(\exists x\) that is a better response for \(\min\) to \(y^*\) i.e. \(x^T C y^* < x^*^T C y^* = v^*\)
    - the existence of such \(x\) violates the fact that the optimum of RHS is \(v^*\) and \(y^*\) is an argmax for RHS
    - similarly the existence of a better response to \(x^*\) by \(\max\) violates that the optimum of LHS is \(v^*\) and \(x^*\) is an argmin for the LHS