Game-Theoretic Formalism

• **Def:** A finite *n-player game* is described by:
  • a set of pure strategies/actions per player: $S_p$
  • a utility/payoff function per player: $u_p : \times_q S_q \to \mathbb{R}$
  • $u_p$: can be thought of as $n$-dimensional tensor

• **Def:** A *randomized/mixed strategy* for player $p$ is any $x_p \in \Delta^{S_p}$
  • assigns probability $x_p(j)$ to each $j \in S_p$
  • i.e. $\Delta^{S_p}$ is the simplex whose vertices are identified with the elements of $S_p$

• **Def:** a player’s *expected utility* is
  • $u_p(x_1, ..., x_n) = \sum_{s \in \times_q S_q} u_p(s) x_1(s_1) \cdot \cdot \cdot x_n(s_n) \equiv u_p \cdot (x_1 \otimes x_2 \otimes \cdots \otimes x_n)$

• A 2-player game can be described by a pair of matrices $(R, C)_{m \times n}$
  • rows $\leftrightarrow S_1$; columns $\leftrightarrow S_2$
  • player 1: “row player”; player 2: “column player”
  • mixed strategies $x \in \Delta^m$ for row player, $y \in \Delta^n$ for column player
  • expected utility of row player: $x^T R y$; expected utility of column player: $x^T C y$
Nash Equilibrium

• **Def:** a collection of mixed strategies $x_1, \ldots, x_n$ is a *Nash equilibrium* iff
  \[
  \forall i, x'_i: \quad u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i})
  \]
  (recall that: if $x_1, \ldots, x_n$ are player strategies, then $x_{-i}$ denotes the strategies of all players except player $i$’s)

• In 2-player games: $(x, y)$ is Nash equilibrium iff
  \[
  \forall x': x^T R y \geq x'^T R y \\
  \forall y': x^T C y \geq x^T C y'
  \]
Nash’s Theorem

[Nash 1950]: Every finite game (i.e. game with a finite number of players and a finite number of pure strategies per player) has a Nash equilibrium.

• **Proof (last time):** using Brouwer’s fixed point theorem.

• **[Brouwer 1911]:** Every continuous function $f: D \to D$ from a convex compact set $D$ to itself has a fixed point, i.e. some $x^* = f(x^*)$. 
Two-player Zero-Sum games

**Minimax Theorem [von Neumann’28]:** Consider a two-player game zero-sum game \((R, C)_{m \times n}\) i.e. \(R + C = 0\). Then

\[
\min_{x \in \Delta^m} \max_{y \in \Delta^n} x^T Cy = \max_{y \in \Delta^n} \min_{x \in \Delta^m} x^T Cy \quad (*)
\]

**Interpretation:**

- \((*)\) says: “If \(\forall y, \exists x \text{ s.t. } x^T Cy \leq v^* \Rightarrow \exists x, \forall y \text{ s.t. } x^T Cy \leq v^*”
- If \(x^*\) is argmin of LHS, \(y^*\) argmax of RHS, \(v^*\) optimal value of \((*)\), then \((x^*, y^*)\) is a Nash equilibrium, i.e. if \(min\) and \(max\) adopt \(x^*\) and \(y^*\) then (i) \(min\) pays \(v^*\) to \(max\) and (ii) no player can improve by unilaterally deviating
- why? Because
  - under \((x^*, y^*)\) \(min\) pays \(max\) at most \(v^*\) (since \(v^*\) optimum of LHS and \(x^*\) is argmin)
  - under \((x^*, y^*)\) \(max\) receives from \(min\) at least \(v^*\) (since \(v^*\) optimum of RHS and \(y^*\) is argmax)
  - by the above two: under \((x^*, y^*)\) \(min\) pays exactly \(v^*\) to \(max\), hence (i) is proven
  - to prove (ii), suppose \(\exists x\) that is a better response for \(min\) to \(y^*\) i.e. \(x^T Cy^* < x^{*T} Cy^* = v^*\)
    - the existence of such \(x\) violates the fact that the optimum of RHS is \(v^*\) and \(y^*\) is an argmax for RHS
    - similarly the existence of a better response to \(x^*\) by \(max\) violates that the optimum of LHS is \(v^*\) and \(x^*\) is an argmin for the LHS
Two-player Zero-Sum games

**Minimax Theorem [von Neumann’28]**: Consider a two-player game zero-sum game \((R, C)_{m \times n}\) i.e. \(R + C = 0\). Then \(\min_{x \in \Delta^m} \max_{y \in \Delta^n} x^T C y = \max_{y \in \Delta^n} \min_{x \in \Delta^m} x^T C y \) (*)&

**Interpretation:**
- (*) says: “If \(\forall y, \exists x\) s.t. \(x^T C y \leq v^* \Rightarrow \exists x, \forall y\) s.t. \(x^T C y \leq v^*\)”
- If \(x^*\) is argmin of LHS, \(y^*\) argmax of RHS, \(v^*\) optimal value of (*), then \((x^*, y^*)\) is a Nash equilibrium, i.e. if \(\min\) and \(\max\) adopt \(x^*\) and \(y^*\) then (i) \(\min\) pays \(v^*\) to \(\max\) and (ii) no player can improve by unilaterally deviating
- thus von Neumann’s theorem establishes the existence of a Nash equilibrium in two-player zero-sum games

**von Neumann:** “As far as I can see, there could be no theory of games ... without that theorem ...
I thought there was nothing worth publishing until the Minimax Theorem was proved”

**Connection to mathematical programming:**
- [von Neumann-Dantzig’47, Adler’13, Brooks-Reny’21]: minimax eq computation \(\Leftrightarrow\) Linear Programming
- Generalizes to convex-concave objectives w/ general convex compact constraint sets
  - In this case, equivalence to convex programming
### Proof of von Neumann’s Minimax Theorem

**Minimax Theorem [von Neumann’28]:** Consider a two-player game zero-sum game \((R, C)_{m \times n}\) i.e. \(R + C = 0\). Then \[
\min_{x \in \Delta^m} \max_{y \in \Delta^n} x^T Cy = \max_{y \in \Delta^n} \min_{x \in \Delta^m} x^T Cy \quad (*)
\]

- Here we’ll do a proof using Strong Linear Programming duality
- **Proof:**

**LHS:**

\[
LP1: \quad \min z \\
x^T C e_j \leq z, \forall j \\
x \in \Delta^m
\]

**RHS:**

\[
LP2: \quad \max w \\
e_i^T C y \geq w, \forall i \\
y \in \Delta^n
\]

LP1 and LP2 are duals!  
Strong LP duality: LP1=LP2
On October 3, 1947, I visited him (von Neumann) for the first time at the Institute for Advanced Study at Princeton.

I remember trying to describe to von Neumann, as I would to an ordinary mortal, the Air Force problem. I began with the formulation of the linear programming model in terms of activities and items, etc.

Von Neumann did something which I believe was uncharacteristic of him. “Get to the point,” he said impatiently. Having at times a somewhat low kindling-point, I said to myself “O.K., if he wants a quicky, then that’s what he will get.”

In under one minute I slapped the geometric and algebraic version of the problem on the blackboard. Von Neumann stood up and said “Oh that!” Then for the next hour and a half, he proceeded to give me a lecture on the mathematical theory of linear programs.

At one point seeing me sitting there with my eyes popping and my mouth open (after I had searched the literature and found nothing), von Neumann said: “I don’t want you to think I am pulling all this out of my sleeve at the spur of the moment like a magician. I have just recently completed a book with Oskar Morgenstern on the theory of games. What I am doing is conjecturing that the two problems are equivalent. The theory that I am outlining for your problem is an analogue to the one we have developed for games.” Thus I learned about Farkas’ Lemma, and about duality for the first time.
Nash Equilibrium Existence: two-player zero-sum games

[von Neumann ’28:]
In two-player zero-sum games, it always exists.
[original proof used fixed point arguments]

[Danzig ’47]
LP duality

[Adler ’13]
[Brooks-Reny’21]

No-regret Learning
cf next week lectures
Nash Equilibrium Existence: general games

[John Nash ’50]: A Nash equilibrium exists in every finite game.

Deep influence in Economics, enabling other existence results.

Proof non-constructive (uses Brouwer’s fixed point theorem)

No simpler proof has been discovered

[Daskalakis-Goldberg-Papadimitriou’06]: no simpler proof exists

i.e. \( \text{Nash Equilibrium} \leftrightarrow \text{Brouwer’s Fixed Point Theorem} \)

cf lectures week after next
Beyond Nash Equilibrium?

Consider other equilibrium concepts
e.g. correlated equilibrium

Consider outcomes of dynamical behavior
e.g. no-regret learning

Data-Set from Microsoft’s Bing
“Econometrics for Learning Agents” [Nekipelov, Syrgkanis, Tardos’15] Stationarity of behavior inconsistent with data-sets
Menu

• Refresher: Nash & von Neumann
• Correlated Equilibrium
Menu

• Refresher: Nash & von Neumann
• Correlated Equilibrium
Consider the following two-player traffic light game:

<table>
<thead>
<tr>
<th></th>
<th>STOP</th>
<th>GO</th>
</tr>
</thead>
<tbody>
<tr>
<td>STOP</td>
<td>(0,0)</td>
<td>(0,1)</td>
</tr>
<tr>
<td>GO</td>
<td>(1,0)</td>
<td>(-100,-100)</td>
</tr>
</tbody>
</table>

Two pure strategy Nash Equilibria: (GO,STOP), and (STOP,GO)
• but under these one player never gets any utility...

One mixed strategy equilibrium: each player plays GO w/ prob \( \frac{1}{101} \) STOP w/ prob \( \frac{100}{101} \)
• results in an accident w/ probability \( \approx 0.01\% \)
• no player goes w/ probability 98%
Traffic Lights

A better outcome would be the following, which is fair, has social welfare 1, and doesn’t risk death:

<table>
<thead>
<tr>
<th></th>
<th>STOP</th>
<th>GO</th>
</tr>
</thead>
<tbody>
<tr>
<td>STOP</td>
<td>0%</td>
<td>50%</td>
</tr>
<tr>
<td>GO</td>
<td>50%</td>
<td>0%</td>
</tr>
</tbody>
</table>

No Nash equilibrium attains the above probabilities
Worse still: *no pair of mixed strategies can attain this distribution over action profiles*

Traffic lights do!
  *by correlating players’ behavior*

Obeying traffic lights is not just a matter of obedience...
  *following the suggestion of the traffic light is a best response!*
Correlated Equilibrium \textbf{[Aumann'74]}

**Def:** A \textit{correlated equilibrium} is a joint distribution $D(s_1, ..., s_n)$ over pure strategy profiles such that for every player $i$, every pair of pure strategies $s_i$ and $s_i'$ s.t. $s_i$ is sampled with non-zero probability by $D$ for player $i$:

$$\mathbb{E}_{s_{-i} \sim D(\cdot | s_i)}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{s_{-i} \sim D(\cdot | s_i)}[u_i(s'_i, s_{-i})]$$

In words: “A distribution $D$ over pure strategy profiles such that after a profile $s$ is drawn from $D$, playing $s_i$ is optimal for player $i$ conditioned on seeing $s_i$ but not seeing $s_{-i}$ and assuming everyone else will play according to $s_{-i}$.”

For example: Conditioned on seeing STOP, you know your opponent will see GO, so STOP is a best response for you assuming opponent will follow the GO recommendation. Conditioned on seeing GO, you know your opponent will see STOP, so GO is a best response assuming opponent will follow the STOP recommendation.
Hierarchies

1. Observe: Nash Equilibria are also Correlated Equilibria — they just correspond to product measures (wherein $s_i$ contains no information about $s_{-i}$).

2. But Correlated Equilibria are a larger/richer set.

3. We can define still larger sets!

**Def:** A *coarse correlated equilibrium* is a distribution $D(s_1, ..., s_n)$ over pure strategy profiles such that for every player $i$ and every pure strategy $s'_i$:

$$
\mathbb{E}_{s \sim D}[u_i(s)] \geq \mathbb{E}_{s \sim D}[u_i(s'_i, s_{-i})]
$$

4. The difference: the recommendation just has to be optimal on average, not *conditioned* on having seen it.

5. Whether it is sensible depends on whether you have to commit to following the correlating “device” $D$ up front, or have the option of deviating after seeing its suggestion.
Hierarchies

Consider game:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(1,1)</td>
<td>(-1,-1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>B</td>
<td>(-1,-1)</td>
<td>(1,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>C</td>
<td>(0,0)</td>
<td>(0,0)</td>
<td>(-1.1,-1.1)</td>
</tr>
</tbody>
</table>

and joint distribution $D$:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1/3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td>1/3</td>
</tr>
</tbody>
</table>

Expected payoff of, say the row player, from committing to follow recommendation of $D$ assuming that the column player does also is $>0$

Expected payoff of row from playing fixed strategy $A$ or $B$ assuming that the column player follows the recommendation of $D$ is: $(1/3) \cdot 1 - (1/3) \cdot 1 + (1/3) \cdot 0 = 0$

Expected payoff of row from playing fixed strategy $C$ assuming that the column player follows the recommendation of $D$ is: less than zero.

Hence this is a CCE even though conditioned on being told to play $C$, it is not a best response. This means that the given distribution is a CCE but not a CE.
Hierarchies

Solution Concept Recap:

\[ DSE \subset^* PSNE \subset MSNE \subset CE \subset CCE \]

1. **Starting at MSNE, we have guaranteed existence.**

2. **Claim:** Starting at CE, we have computational tractability.
   - First, can write CE as a linear program - *today*
   - Second, no-regret learning converges to CCE in general (and with extra work can define no-regret learner that converge to CE) – *next week*

**Key**

- **DSE**: dominant strategy equilibrium
- **PSNE**: pure Nash equilibrium
- **MSNE**: (mixed) Nash equilibrium
- **CE**: correlated equilibrium
- **CCE**: coarse correlated equilibrium

\[ \subset \]: strict inclusion for some games
\[ \subset^* \]: inclusion for games w/ no identical-payoff strategies;
otherwise \( DSE \subset MSNE \)
Menu
• Refresher: Nash & von Neumann
• Correlated Equilibrium
Menu

- Refresher: Nash & von Neumann
- Correlated Equilibrium
  - computation
**Def:** A *correlated equilibrium* is a joint distribution $D(s_1, ..., s_n)$ over pure strategy profiles such that for every player $i$, every pair of pure strategies $s_i$ and $s_i'$ s.t. $s_i$ is sampled with non-zero probability by $D$ for player $i$:

$$
\mathbb{E}_{s_i \sim D(\cdot|s_i)}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{s_i \sim D(\cdot|s_i)}[u_i(s_i', s_{-i})]
$$

\[
\begin{align*}
\text{max } & 0 \\
\text{s.t. } & \forall i, \forall s_i, s_i': \sum_{s_{-i}} D(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} D(s_i, s_{-i})u_i(s_i', s_{-i}) \\
& \forall s: D(s) \geq 0 \\
& \sum_s D(s) = 1
\end{align*}
\]

N.B. first constraint same as

$$
\forall i, \forall s_i, s_i': \quad D(s_i) \sum_{s_{-i}} D(s_{-i}|s_i)u_i(s_i, s_{-i}) \geq D(s_i) \sum_{s_{-i}} D(s_{-i}|s_i)u_i(s_i', s_{-i})
\]

$$
\Leftrightarrow \forall i, \forall s_i \text{ s.t. } D(s_i) > 0, \forall s_i': \quad \sum_{s_{-i}} D(s_{-i}|s_i)u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} D(s_{-i}|s_i)u_i(s_i', s_{-i})
\]
Linear Programming Formulation

**Def:** A *correlated equilibrium* is a joint distribution $D(s_1, \ldots, s_n)$ over pure strategy profiles such that for every player $i$, every pair of pure strategies $s_i$ and $s_i'$ s.t. $s_i$ is sampled with non-zero probability by $D$ for player $i$:

$$\mathbb{E}_{s_i \sim D(\cdot | s_i)}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{s_i \sim D(\cdot | s_i)}[u_i(s_i', s_{-i})]$$

\[
\begin{align*}
\max \text{ any linear function of } D \\
\text{s.t. } i, \forall s, s_i': \\
\forall s: & \quad D(s) \geq 0 \\
\forall s: & \quad \sum_s D(s) = 1 \\
\forall s_i, s_i': \quad & \sum_{s_{-i}} D(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} D(s_i', s_{-i}) u_i(s_i', s_{-i}) \\
\iff \forall i, \forall s_i, s_i': \quad & D(s_i) \sum_{s_{-i}} D(s_{-i} | s_i) u_i(s_i, s_{-i}) \geq D(s_i) \sum_{s_{-i}} D(s_{-i} | s_i) u_i(s_i', s_{-i}) \\
\text{N.B. first constraint same as } \iff \forall i, \forall s_i \text{ s.t. } D(s_i) > 0, \forall s_i': \quad & \sum_{s_{-i}} D(s_{-i} | s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} D(s_{-i} | s_i) u_i(s_i', s_{-i})
\end{align*}
\]
Def: A **correlated equilibrium** is a joint distribution $D(s_1, \ldots, s_n)$ over pure strategy profiles such that for every player $i$, every pair of pure strategies $s_i$ and $s_i'$ s.t. $s_i$ is sampled with non-zero probability by $D$ for player $i$:

$$\mathbb{E}_{s_i \sim D(s_i)}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{s_i \sim D(s_i)}[u_i(s_i', s_{-i})]$$

\[\begin{align*}
\text{max} & \quad \sum_s D(s) \sum_i u_i(s) \\
\text{s.t.} & \quad \forall i, \forall s_i, s_i': \quad \sum_{s_{-i}} D(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} D(s_i, s_{-i}) u_i(s_i', s_{-i}) \\
& \quad \forall s: \quad D(s) \geq 0 \\
& \quad \sum_s D(s) = 1
\end{align*}\]

N.B. first constraint same as

$$\forall i, \forall s_i, s_i': \quad D(s_i) \sum_{s_{-i}} D(s_{-i}|s_i) u_i(s_i, s_{-i}) \geq D(s_i) \sum_{s_{-i}} D(s_{-i}|s_i) u_i(s_i', s_{-i})$$

$$\iff \forall i, \forall s_i \text{ s.t. } D(s_i) > 0, \forall s_i': \quad \sum_{s_{-i}} D(s_{-i}|s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} D(s_{-i}|s_i) u_i(s_i', s_{-i})$$
**Def:** A *correlated equilibrium* is a joint distribution $D(s_1, \ldots, s_n)$ over pure strategy profiles such that for every player $i$, every pair of pure strategies $s_i$ and $s_i'$ s.t. $s_i$ is sampled with non-zero probability by $D$ for player $i$:

$$
\mathbb{E}_{s_i \sim D(\cdot|s_i)}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{s_i \sim D(\cdot|s_i)}[u_i(s_i', s_{-i})]
$$

\[
\begin{align*}
\text{max} \sum_s D(s)u_{i^*}(s) & \quad \text{(welfare of a particular player $i^*$)} \\
\text{s.t.} & \quad \forall i, \forall s_i, s_i': \sum_{s_{-i}} D(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} D(s_i, s_{-i})u_i(s_i', s_{-i}) \\
& \quad \forall s: D(s) \geq 0 \\
& \quad \sum_s D(s) = 1
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\]

N.B. first constraint same as

$$
\forall i, \forall s_i, s_i': D(s_i) \sum_{s_{-i}} D(s_{-i}|s_i)u_i(s_i, s_{-i}) \geq D(s_i) \sum_{s_{-i}} D(s_{-i}|s_i)u_i(s_i', s_{-i})
$$

$\Leftrightarrow \forall i, \forall s_i$ s.t. $D(s_i) > 0, \forall s_i':$

$$
\sum_{s_{-i}} D(s_{-i}|s_i)u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} D(s_{-i}|s_i)u_i(s_i', s_{-i})
$$
N.B.

• LP has as many variables as there are pure strategy profiles
  • $n$ players, $s$ strategies each: $s^n$
  • so to solve this LP need time $\text{poly}(s^n)$
  • Technically, this is polynomial in the description of the game, as every utility tensor has as many entries as there are variables in the LP

• Can we do better?
  • Yes, if we are OK with approximate CE/CCE computation
    • using no-regret learning (next week!) can find CCE/CE in time roughly: $\frac{\text{poly}(s,n)}{\varepsilon^2}$
  • Yes, if the game has more structure,
    • e.g. graphical game (every player’s utility depends on $d$ other players, where $d << n$)
    • use “Ellipsoid Against Hope” algorithm of [Papadimitriou’05]