University of Pennsylvania

ESE 500 Mid-Term Exam
10/23/03

Exam Rules

- Exam is open-book in the following sense: consulting your text book or other references is allowed, before you start writing down your solutions. However, during the writing you are not allowed to use any references.
- **Collaboration is strictly forbidden.**
- The exam is due on **Monday October 27th at 12:00pm (EST)**. **No late exams are accepted**. You may leave the exam with my administrative assistant Ms.Dru Spanner. If you can not come to my office on Thursday, you can Fax/Email your solutions to me (Fax:215-573-2068).
- The exam consists of 5 problems, 20 points each, for a total of 100 points.

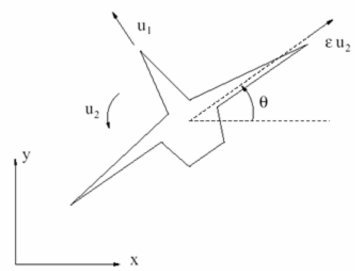

Fig. 1. The VTOL aircraft.

1. The following is a planar model of a Vertical Take-off and Landing (VTOL) aircraft such as McDonald Douglas' Harrier AV-F8 or Lockheed's F35 Joint Strike fighter around hover (cf. Figure 1):

$$\begin{aligned} m\ddot{x} &= -u_1 \sin\theta + \epsilon u_2 \cos\theta \\ m\ddot{y} &= u_1 \cos\theta + \epsilon u_2 \sin\theta - mg \\ J\ddot{\theta} &= u_2, \end{aligned}$$

where x, y are the position of the center of mass of the aircraft in the vertical plane and θ is the roll angle of the aircraft. u_1 and u_2 are the thrust forces (control inputs). The thrust is generated by a powerful fan and is vectored into two forces u_1 and u_2. J is the moment of inertia, and ϵ is a small coupling constant. Find the linearization of this model around the equilibrium solution

$$\tilde{x}(t), \tilde{y}(t), \tilde{\theta}(t) = 0,\ \tilde{u}_1(t) = mg;\ \tilde{u}_2(t) = 0.$$

The linearized model should be time invariant. Without performing any calculations, determine the stability of the linearized model.

2. A matrix is called *skew symmetric*, if its transpose is equal to its negative, i.e., $A^T = -A$. Show that if the $n \times n$ matrix A is skew symmetric, then the solution of the linear system

$$\dot{x}(t) = Ax(t)$$

with $x(0) = x_0$ satisfies

$$||x(t)||_2 = ||x_0||_2 \qquad \forall\, t \geq 0,$$

where $|| \cdot ||_2$ is the Euclidean norm of a vector.

3. Suppose M is an $n \times n$ invertible matrix with distinct eigenvalues. Show that there exists a possibly complex $n \times n$ matrix R such that

$$M = e^R.$$

4. (a) Show that the solution to the matrix differential equation

$$\dot{X} = AX + XF, \qquad X(0) = C$$

where A and F are given $n \times n$ matrices is

$$X = e^{At} C e^{Ft}.$$

(b) Given a continuous $n \times n$ matrix $A(t)$ which commutes with its integral (i.e., $A(t) \int_{t_0}^{t} A(\sigma)\, d\sigma = (\int_{t_0}^{t} A(\sigma)\, d\sigma) A(t)$) and a constant $n \times n$ matrix F, show how to find a coordinate transformation matrix $P(t)$ with $P(0) = P_0$ that transforms the state equation

$$\dot{x}(t) = A(t)x(t)$$

into

$$\dot{z}(t) = Fz(t).$$

Use the result from the first part as a clue to find the transformation explicitly.

5. consider the Time Invariant linear system

$$\dot{x} = Ax, \qquad y = Cx$$

where $x(0) = x_0$, A is $n \times n$ and all of its eigenvalues have negative real parts, and C is an invertible $n \times n$ output matrix. Show that

$$\int_0^\infty ||y(t)||^2 dt = x_0^T Q x_0$$

where Q is the positive definite solution of

$$A^T Q + QA + C^T C = 0.$$