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Consensus formation under bounded confidence

Jan Christian Dittmer

[a] *Fachbereich Mathematik und Informatik, Universität Bremen, Bremen, Germany*

Abstract

Consensus formation among n experts is modeled as a positive discrete dynamical system in n dimensions. Experts revise their opinion by repeated averaging over the opinions of those experts whom they trust within some range of confidence (BC model). We present a necessary and sufficient condition for reaching a consensus in the uniform BC model. We also extend the BC model to the hierarchic BC model to model hierarchic structures inside the group of experts.

Key words: consensus formation, bounded confidence, repeated averaging, discrete dynamical system, hierarchical structure

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1 The BC Model

Consider a group of experts who have to make a joint assessment of a certain magnitude. Before they meet each of the experts has his own opinion. But being informed about the opinions of all the other experts they may revise their own. These revisions may lead to further revisions by the same reason. The question is whether this iterative process of changing opinions will lead to an agreement, a consensus among the experts, concerning the value of the magnitude.

Denote by x_i^t the opinon of expert $i \in \{1, \ldots, n\}$ at time $t \in \mathbb{N}$. A vector $x \in (\mathbb{R}_{\geq 0})^n$ is called an *opinon profile*. Suppose that expert i arrives at a revision x_i^{t+1} by taking into account the assessments x_j^t of all experts j whose opinions are not too far away from his own opinion; that is, i takes the opinion

Email address: jdittmer@uni-bremen.de (Jan Christian Dittmer).

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of j into account iff $|x_i - x_j| \leq \varepsilon_i$ where $\varepsilon_i > 0$ is a certain level of confidence employed by expert i. This *range of confidence* is supposed to be the same for all experts in the uniform BC model, that is $\varepsilon_i \equiv \varepsilon$.

Definition 1 *For expert i, $1 \leq i \leq n$, and the opinion profile $x \in (\mathbb{R}_{\geq 0})^n$,*

$$I_x(i) = \{j = 1, \ldots, n \;:\; |x_i - x_j| \leq \varepsilon\} \tag{1}$$

is called the set of trusted experts *of expert i.*

Each expert revises his opinion by taking the arithmetic mean of all those experts inside the range of confidence. So we write the revision of an opinion profile x as matrix multiplication $A(x)x$ where $A(x)$ is the weight matrix and a_{ij} the weight expert i gives to expert j.

Definition 2 *Let $m_i = \#I_x(i)$. The $n \times n$-matrix $A(x)$ defined by*

$$a_{ij}(x) = \begin{cases} \frac{1}{m_i} & , \textit{if } j \in I_x(i) \\ 0 & \textit{otherwise} \end{cases} \tag{2}$$

is called transition matrix *of x.*

This definition implies that $A(x)$ is row-stochastic.

The uniform BC (bounded confidence) model is defined as a discrete dynamical system with

$$x^{t+1} = A(x^t)x^t \tag{3}$$

and $x^0 \in (\mathbb{R}_{\geq 0})^n$, where $A(x^t)$ is the transition matrix of x^t:

A consensus has been reached at time t_1 if there is a value $u \in \mathbb{R}_{\geq 0}$ such that $x_i^t \equiv u$ for all i and for $t \geq t_1$.

2 Convergence to Consensus in the uniform BC model

To examine the model for reaching a consensus we relabel the experts such that $x_1 \leq x_2 \leq \ldots \leq x_n$. It is easy to show that such an ordered opinion profile will not loose its order in the next time step. So we can use the same order for all time steps and the question of consensus is independent of relabeling. If each expert trusts at least his neighbours then the opinion profile x is called an ε-chain, that is $x_{i+1} - x_i \leq \varepsilon$ for all $1 \leq i \leq n-1$. If $x_{i+1} - x_i > \varepsilon$ for some i we speak of a crack between expert $i+1$ and expert i.

If there is a crack between two experts at time t then there will be a crack between them at time $t+1$. So for reaching a consensus it is necessary that there is no crack in the profile at any time step. (cf. Krause (2000)).

Below we show that this condition is also sufficient.

Proposition 3 *If the opinion profile is an ε-chain, the diagonal and both off-diagonals of the transition matrix are positive.*

It is easy to show that the product of $n-1$ such matrices A_i is positive, that is $B = A_{n-1} \cdot \ldots \cdot A_0 > 0$ ($\iff$ $b_{ij} > 0$ for all i and j).

We can rewrite system (3) for time steps in $(n-1) \cdot \mathbb{N}$ as

$$\tilde{x}^t := x^{(n-1)(t+1)} = B_t B_{t-1} \ldots B_0 x^0 \tag{4}$$

where $B_k = A_{(n-1)(k+1)-1} \cdots A_{(n-1)k}$, $A_j = A(x^j$ and the $B_k's$ are all positive.

Furthermore, the product of transition matrices is row-stochastic.

To measure the progress in reaching a consensus we define the *range* v of a opinion profile x as $\mathrm{v}(x) = \max_i x_i - \min_i x_i = \max_{1\le i,j\le n} (x_i - x_j)$.

Thus, a consensus corresponds to $\mathrm{v}(x) = 0$.

The following Lemma is well-known in literature (cf. Krause (2000)) for the continuous function v.

Lemma 4 *If A is a row-stochastic matrix then*

$$\mathrm{v}(Ax) \le \left(1 - \min_{1\le i,j\le n} \sum_{k=1}^{n} \min\{a_{ik}, a_{jk}\}\right) \mathrm{v}(x) \tag{5}$$

for all $x \in (\mathbb{R}_{\ge 0})^n$.

Corollary 5 *The sequence $(\mathrm{v}(x^t))_t$ is monoton decreasing and bounded from below by 0.*

To proof the convergence of this sequence it is sufficient to find a converging subsequence.

For the positive row-stochastic matrix B_t the Lemma implies $\mathrm{v}(B_t\tilde{x}^t) < \mathrm{v}(\tilde{x}^t)$ for all $\tilde{x} \in (\mathbb{R}_{\ge 0})^n$.

Lemma 6 $\mathrm{v}(B_t\tilde{x}^t) \le q\mathrm{v}(\tilde{x}^t)$ *for some $q < 1$.*

Corollary 7 *The sequence $(\mathrm{v}(\tilde{x}^t))_t$ converges to 0.*

Theorem 8 *A consensus will be reached iff the opinion profile x^t is an ε-chain for every $t \in \mathbb{N}$. This consensus will be reached in finite time.*

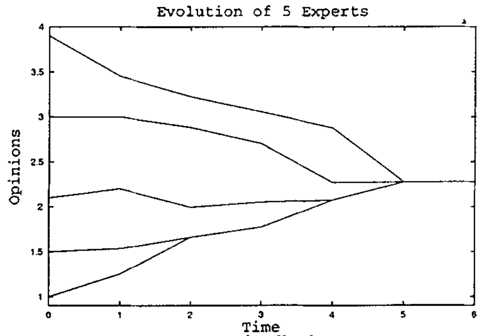
PROOF. Corollary 7 shows that $(\mathrm{v}(\tilde{x}^t))_t$ is a converging subsequence of $(\mathrm{v}(x^t))_t$, which is monoton decreasing and bounded from below by 0 (Corollary 5). The limit point of $(\mathrm{v}(\tilde{x}^t))_t$ is 0. So $(\mathrm{v}(x^t))_t$ converges to 0, which means a consensus is reached.

Once the range of x^t is below ε a consensus will be reached in the next time step because $I_x(i) = \{1, \ldots, n\}$ for every expert i, that is every expert will revise his opinion by taking the average of all the others opinions. So consensus will be reached in finite time. □

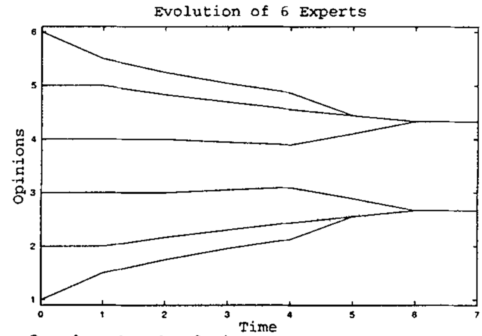
Theorem 9 *The BC model reaches a fixed point in finite time.*

PROOF. Once a crack appears in the profile the experts are divided into independent groups. Each new crack will refine the partition of experts. This process of refinement has to stop due to the finiteness of the number of experts. From this time on the opinion profiles of each seperated group of experts will be an ε-chain. So each subgroup will reach a consensus in finite time and this consensus paattern provides a fixed point. □

Examples with range of confidence $\varepsilon = 1$



consensus in 5 timesteps



fixed point in 6 time steps – no consensus

3 Generalisation: A hierarchic BC model

Imagine a number of experts with a hierarchy among them, that is an expert is subordinated, superior, indifferent or of equal status to another expert We can gather all those experts at the same level in this hierarchy and will call this group of experts *h-group*. Each expert is a member of exactly one h-group, that is the set of h-groups is a partition of all experts. And we have an order between these h-groups which is not necessarily linear. Below we restrict the generality by the demand for the existence of a maximum in the hierarchy.

More precisely, we define a hierarchy as follows.

Definition 10 *A hierarchy $\mathcal{H}$ among n experts is an ordered set. The elements of this set are a partition of the set $\{1, \ldots, n\}$ and called h-groups. Furthermore there exists a maximum S in $\mathcal{H}$.*

If expert i is trusted by expert j from a subordinated h-group, expert i is called an idol *of j.*

For a *hierarchic transition matrix A* should hold:

- Each expert (not out of the primary h-group S) should have an idol.
- A h-group has an idol in any directly superior h-group

- An expert trusts only experts of superior h-groups and experts in his own h-group.
- A is row-stochastic.
- An expert is not only trusting his idols. (At least a little bit of self-confidence)
- The experts of one h-group trust each other by a local (uniform) BC model.

Example for a hierarchic transition matrix

$$\left[\begin{array}{ccc|ccc|c}
\multicolumn{3}{c|}{\textit{local BC model}} & \multicolumn{3}{c|}{0} & 0 \\
\hline
1/4 & 0 & 0 & \multicolumn{3}{c|}{\textit{scaled}} & \\
1/8 & 1/8 & 0 & \multicolumn{3}{c|}{\textit{local}} & 0 \\
0 & 0 & 1/2 & \multicolumn{3}{c|}{\textit{BC model}} & \\
\hline
0 & 0 & 0 & 0 & 0 & 1/3 & \\
0 & 0 & 0 & 0 & 0 & 1/3 & \textit{scaled} \\
0 & 0 & 0 & 0 & 0 & 1/5 & \textit{local} \\
0 & 0 & 1/4 & 0 & 0 & 0 & \textit{BC model} \\
0 & 1/4 & 0 & 0 & 0 & 0 & \\
1/6 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right]$$

There are 3 h-groups: $G_1 = \{1, 2, 3\}$, $G_2 = \{4, 5, 6\}$ and $G_3 = \{7, 8, 9, 10, 11, 12\}$. G_1 is the primary group S. G_3 is subordinated to G_1 and to G_2.

A consensus of all experts implies a consensus of S which is equivalent, as we have learned from the BC model, to the condition that the profile of S (after relabeling) allways is an ε-chain. So this condition is a **necessary criteron** for consensus.

The above examples suggests the following notation for a hierarchic transition matrix

$$A = H + F(H)W(x) = A(x, H)$$

Where the matrix H contains the hierarchic or idol weights in the subdiagonal blocks. $W(x)$ is a block diagonal matrix containing the weights calculated blockwise by the ordinary BC model. $F(H) \in \mathbb{R}^{n \times n}$ is a scaling factor to obtain a row-stochastic matrix A.

Now we can write the hierarchic BC model as a discrete dynamical system

$$x^{k+1} = A(x^k, H_k)x^k$$

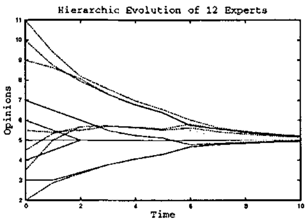
where x^0 is the initial profile and $(H_k)_k$ is a sequence of matrices of hierarchic weights for the given hierarchy.

Theorem 11 *In the hierarchic BC model with constant hierarchic weights a consensus will be reached iff the profile of the primary h-group S is an ε-chain at every time.*

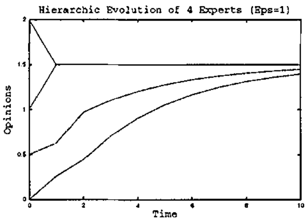
Theorem 12 *In the hierarchic BC model with n experts a consensus will be* *reached if S reaches a consensus and the following condition holds:*

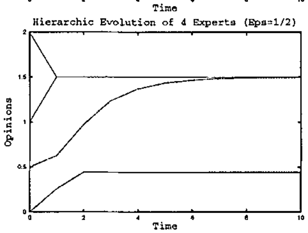
$$\sum_{k=0}^{\infty} \min_{i=1,\dots,n} \sum_{j\in S} (A_k)_{ij} = \infty$$

Examples



Configuration with initial opinion profile 4, 5, 6, 2, 3, 7, $7/2$, $9/2$, $11/2$, 9, 10, 11 and constant hierarchic weights as defined above in the expample of a hierarchic transition matrix. A consensus will be reached.





Harderning of position: Exponential decreasing of idol weight of one expert. The two examples with different range of confidence in the subordinated group show that the condition in Theorem 12 is not necessary. In the first example the ε-chain in the subordinated h-group leads to a consensus of all experts. In the second example the range of confidence is not strong enough to get the expert with inital opinion 0 into consensus.

References

Chatterjee, S., 1975. Reaching a consensus: Some limit theorems. Proc. Int. Statist. Inst. , 159–164.

Chatterjee, S., Seneta, E., 1977. Toward consensus: some convergence theorems on repeated averaging. J. Appl. Prob 14, 89–97.

Cohen, J., Hajnal, J., Newman, C., 1986. Approaching consensus can be delicate when positions hardens. Stochastic Proc. and Appl. 22, 315–322.

DeGroot, M. H., 1974. Reaching a consensus. Journal of American Statistical Association 69 (345), 118–121.

French, J. R. P., 1956. A formal theory of social power. Psychological Review 63 (3).

Hegselmann, R., Flache, A., Möller, V., 1998. Cellular automata models of solidarity and opinion formation: sensitivity analysis. Universität Bayreuth , preprint.

Krause, U., 2000. A discrete nonlinear and non-autonomous model of consensus formation. Communications in Difference Equations (Eds. S. Elaydi, G. Ladas, J. Popenda, J. Rakowski), Gordon and Breach Science Publishers, Amsterdam , 227–236.

Krause, U., Nesemann, T., 1999. Differenzengleichungen und diskrete dynamische Systeme. B.G. Teubner Stuttgart.

Lehrer, K., Wagner, C., 1981. Rational Consensus in Science and Society. D. Reidel Publishing Company.

Stone, M., 1961. The opinion pool. Annals of Mathematical Statistics 32, 1339–42.