Guaranteed-Cost Design of Continuous-Time Takagi-Sugeno Fuzzy Controllers via Linear Matrix Inequalities

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Abstract

In this paper, we extend some recent results regarding the synthesis of Takagi-Sugeno (T-S) fuzzy controllers via Linear Matrix Inequalities Using a guaranteed-cost approach. This method not only guarantees stability, but also minimizes an upper bound on a linear quadratic performance measure. A numerical example is presented to show the effectiveness of this method

Key Words: Fuzzy systems, Takagi-Sugeno controllers, Linear Matrix Inequalities, guaranteed-cost design.

1. Introduction

Recently there has been a great deal of interest in using Takagi-Sugeno fuzzy models to approximate nonlinear systems. This interest relies on the fact that dynamic T-S models are easily obtained by linearization of a the nonlinear plant around different operating points. Once the T-S fuzzy models are obtained, linear control methodology can be used to design local state feedback controllers for each linear model. Aggregation of the fuzzy rules results in a generally nonlinear model, but in a very special form, which is exactly the same as a time varying and/or nonlinear system described by a set of Polytopic Linear Differential Inclusions (PLDI's) [1]. Since powerful convex optimization algorithms exist for dealing with these kind of systems, it is natural to use these algorithms for design of stabilizing T-S fuzzy controllers.(See [8, 9] and the references therein). Sufficient conditions for the stability of T-S systems were first proposed in [6]. These sufficient conditions required the existence of a common positive-definite matrix P which would satisfy a set of Lyapunov inequalities. Although looking for a common positive-definite solution of the Lyapunov inequalities is by no means easy, the Lyapunov inequalities can be transformed into a set of Linear Matrix Ineaqualities (LMI's from now on), which can be efficiently solved in polynomial time using the recently developed interior-point methods [5]. The stability of these systems have been discussed in detail in the literature [8], but there have been few results that have gone beyond stability, and have considered performance. Authors in [9] have added the degree of stability, and have shown that controller design with guaranteed degree of stability can be transformed into a Generalized EigenValue Problem (GEVP) [1]. Recently the authors in [7] have added an LMI condition which guarantees that the control action is norm-bounded. In this paper, we generalize these results to the problem of minimizing the

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expected value of a quadratic performance measure with respect to randomized initial conditions, with zero mean and a covariance equal to the identity. Using the guaranteed-cost approach [2], we minimize an upper bound on an LQ performance measure representing the control effort and the regulation error. We show that this problem can be transformed into a trace minimzation problem, which can then be solved using any of the available convex optimization software packages.

The structure of this paper is as follows: In section (2), we present an overview of dynamic Takagi Sugeno systems and LMI formulation of their stability conditions. Section (3) deals with the guaranteed-cost performance problem and an upper bound on the performance measure and its formulaion as LMIs. Numerical examples are presented in section (4). Finally conclusions and some future work are discussed in section (5).

2. Dynamic Takagi-Sugeno Fuzzy Models

A dynamic T-S fuzzy model is described by a set of fuzzy "IF... THEN" rules with fuzzy sets in the antecedents and dynamic LTI systems in the consequents. A generic T-S plant rule can be written as follows:

i^{th} Plant Rule: IF $x_1(t)$ is $\tilde{M}_{i1}$ and $\ldots, x_n(t)$ is $\tilde{M}_{in}$ THEN $\dot{x} = A_i x + B_i u$

where $x \in \mathrm{I\!R}^{n\times 1}$ is the state vector, $i = \{1, \cdots, r\}$, r is the number of rules, $\tilde{M}_{ij}$ are input fuzzy sets, $A_i \in \mathrm{I\!R}^{n\times n}, B_i \in \mathrm{I\!R}^{n\times m}$, and $u \in \mathrm{I\!R}^{m\times 1}$.
Using singleton fuzzifier, max-product inference and center average defuzzifier, we can write the aggregated fuzzy model as

$$\dot{x} = \frac{\sum_{i=1}^{r} w_i(x)(A_i x + B_i u)}{\sum_{i=1}^{r} w_i(x)} \tag{1}$$

where w_i is defined as

$$w_i(x) = \prod_{j=1}^{n} \mu_{ij}(x_j) \tag{2}$$

where μ_{ij} is the membership function of jth fuzzy set in the ith rule. Defining

$$\alpha_i(x) = \frac{w_i(x)}{\sum_{i=1}^{r} w_i(x)} \tag{3}$$

we can write (1) as

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x)(A_i x + B_i u) \tag{4}$$

where $\alpha_i > 0$ and $\sum_{i=1}^{r} \alpha_i = 1$
Using the same method for generating T-S fuzzy rules for the controller, we have
i^{th}Controller Rule: IF $x_1(t)$ is M_{i1} and $\ldots x_n(t)$ is M_{in} THEN $u = -K_i x$
The over all controller would be

$$u = -\sum_{i=1}^{r} \alpha_i(x) K_i x \tag{5}$$

Replacing (5) in (4), we obtain the following equation for the closed loop system:

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(x)\alpha_j(x)(A_i - B_i K_j)x \tag{6}$$

We have the following theorem for stability

Theorem 1 *[8]:The closed loop fuzzy system (6) is globally asymptotically stable if there exist a common positive definite matrix P which satisfies the following Lyapunov inequalities:*

$$\begin{aligned} (A_i - B_i K_i)^T P + P(A_i - B_i K_i) &< 0 \\ G_{ij}^T P + P G_{ij} &< 0 \\ i = 1, \cdots, r \quad j < i \le r & \end{aligned} \tag{7}$$

where G_{ij} is defined as

$$G_{ij} = A_i - B_i K_j + A_j - B_j K_i \tag{8}$$

Pre-multiplying and post-multiplying both sides of the inequalities in (7) by P^{-1} and using the following change of variables

$$\begin{aligned} Y &= P^{-1} \\ X_i &= K_i Y \end{aligned} \tag{9}$$

we obtain the following LMIs [1]:

$$\begin{aligned} Y A_i^T + A_i Y - B_i X_i - X_i^T B_i^T &< 0 \\ Y(A_i + A_j)^T + (A_i + A_j)Y - M_{ij} - M_{ij}^T &< 0 \\ i = 1, \cdots, r \quad j < i \le r & \end{aligned} \tag{10}$$

where M_{ij} is defined as:

$$M_{ij} = B_i X_j + B_j X_i \tag{11}$$

If the above LMIs have a common positive-definite solution, stability is guaranteed, but in most practical problems stability by itself is not enough, and the controller needs tosatisfy certain design objectives. This will be discussed in the next section.

3. Guaranteed-Cost Design Using LMIs

In this section we try to achieve performance using a guaranteed cost approach [2]. It is a well know result from the LQR theory that the problem of minimizing the cost function

$$J = \int_0^\infty (x^T Q x + u^T R u) dt \tag{12}$$

subject to:

$$\dot{x} = Ax + Bu \qquad u = -Kx \tag{13}$$

results in finding a positive definite solution of the following Lyapunov equation.

$$(A - BK)^T P + P(A - BK) + Q + K^T R K = 0$$

where $Q \geq 0$ and $R > 0$. We can write the minimum cost J as [2]:

$$J = x_0^T P x_0$$

and P is the solution of the Lyapunov equations. If we write the Lyapunov equation as a matrix inequality instead of an equality, the solution of the inequality will be an upper bound on the performance measure J, and we can reach J by minimizing that upper bound. While this result holds for a single LTI system, we can extend it to the case of equation (6). To avoid the dependency of the cost function J on initial conditions, we assume the initial conditions are random variables with zero mean and a covariance equal to the identity, i.e.,

$$\begin{aligned} \mathbb{E}\{x_0 x_0^T\} &= I \\ \mathbb{E}\{x_0\} &= 0 \end{aligned} \tag{14}$$

where $\mathbb{E}$ is the expectation operator. Our objective is to minimize the expected value of the performance index J with respect to all possible initial conditions with zero mean and covariance equal to the identity. Now we can state the following lemma:

Lemma 1 *For random initial conditions with zero mean and covariance equal to the identity, we have*

$$\mathbb{E}_{x_0}\{x_0^T P x_0\} = tr(P) \tag{15}$$

Using the above lemma, we can state the following theorem:

Theorem 2 *Consider the closed loop fuzzy system (6). Then, we have the following bound on J:*

$$J = \mathbb{E}_{x_0} \int_0^\infty (x^T Q x + u^T R u) dt < tr(P) \tag{16}$$

and P is the positive-definite solution of the following inequalities:

$$\begin{aligned} H_i^T P + P H_i + Q + \sum_{i=1}^r K_i^T R K_i &< 0 \\ G_{ij}^T P + P G_{ij} + Q + \sum_{i=1}^r K_i^T R K_i &< 0 \\ i = 1, \cdots, r \quad j < i \leq r \end{aligned} \tag{17}$$

where u is defined in equation (5), M_{ij} is the same as in (8), and H_i is defined as:

$$H_i = A_i - B_i K_i.$$

Proof: We already know that $J < tr(P)$ where P satisfies the following inequalities:

$$\begin{aligned} H_i^T P + P H_i + Q + (\sum_{i=1}^r \alpha_i K_i)^T R (\sum_{i=1}^r \alpha_i K_i) &< 0 \\ G_{ij}^T P + P G_{ij} + Q + (\sum_{i=1}^r \alpha_i K_i)^T R (\sum_{i=1}^r \alpha_i K_i) &< 0 \\ i = 1, \cdots, r \quad j < i \leq r \end{aligned} \tag{18}$$

We just need to show that

$$(\sum_{i=1}^r \alpha_i K_i)^T R (\sum_{i=1}^r \alpha_i K_i) < \sum_{i=1}^r K_i^T R K_i \tag{19}$$

For simplicity, we will show that the above inequality is true for the case where we only have two rules for the controller, the extension to more than two rules can be done via induction. We need to show that

$$(\sum_{i=1}^2 \alpha_i K_i)^T R (\sum_{i=1}^2 \alpha_i K_i) < \sum_{i=1}^2 K_i^T R K_i \tag{20}$$

To illustrate this, we rewrite the left hand side of the above equation as the following quadratic form:

$$\left[\begin{array}{cc} K_1^T R^{1/2} & K_2^T R^{1/2} \end{array}\right] U \left[\begin{array}{cc} R^{1/2} K_1 & R^{1/2} K_2 \end{array}\right] \quad (21)$$

Where

$$U = \left[\begin{array}{cc} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{array}\right]$$

The right hand side of the equation can be written as:

$$\left[\begin{array}{cc} K_1^T R^{1/2} & K_2^T R^{1/2} \end{array}\right] I_2 \left[\begin{array}{cc} R^{1/2} K_1 & R^{1/2} K_2 \end{array}\right] \quad (22)$$

Where I_2 is the 2×2 identity matrix. To prove the theorem we have to show that

$$\left[\begin{array}{cc} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{array}\right] < \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \quad (23)$$

To complete the proof, we have to show that the difference of the two matrices is positive definite, i.e., we have the following:

$$\left[\begin{array}{cc} 1-\alpha_1^2 & -\alpha_1\alpha_2 \\ -\alpha_1\alpha_2 & 1-\alpha_2^2 \end{array}\right] > 0 \quad (24)$$

This is already satisfied since the nested principal minors are positive, i.e.:

$$1-\alpha_1^2 > 0$$
$$(1-\alpha_1^2)(1-\alpha_2^2) - \alpha_1^2\alpha_2^2 = 1 - (\alpha_1^2 + \alpha_2^2) > 0 \quad (25)$$

The above inequalities are immediate result of the fact that:

$$\begin{aligned} \alpha_i &> 0 \quad i = 1, \cdots, r \\ \sum_{i=1}^{r} \alpha_i &= 1 \end{aligned} \quad (26)$$

■

For later reference, we define

$$\begin{aligned} N_i &= Y A_i^T + A_i Y - B_i X_i - X_i^T B_i^T \quad (27) \\ O_{ij} &= Y(A_i + A_j)^T + (A_i + A_j) Y - M_{ij} - M_{ij}^T \end{aligned}$$

Now, using the same change of variables as (9), and multiplying both sides of equation (17) by P^{-1} and also using theorem (1),we can write (17)as the following inequalities:

$$N_i + YQY + \sum_{i=1}^{r} X_i^T R X_i < 0$$

$$O_{ij} + YQY + \sum_{i=1}^{r} X_i^T R X_i < 0$$
$$i = 1, \cdots, r \ j < i \leq r \quad (28)$$

Using the LMI Lemma [2], we can write the above inequalities as follows:

$$\left[\begin{array}{ccccc} N_i & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2} Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2} X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2} X_r & 0 & 0 & \cdots & -I_{m\times m} \end{array}\right] < 0$$
$$\left[\begin{array}{ccccc} O_{ij} & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2} Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2} X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2} X_r & 0 & 0 & \cdots & -I_{m\times m} \end{array}\right] < 0$$
$$i = 1, \cdots, r \quad j < i \leq r$$
$$(29)$$

To obtain the least possible upper bound using a quadratic Lyapunov function, we have the following optimization problem

Min $tr(Y^{-1})$
Subject To: LMIs in(29)
This is a convex optimization problem which can be solved in polynomial time [5] using any of the available LMI toolboxes. To make it possible to use the MATLAB® LMI Toolbox, we introduce an artificial variable Z, which is an upper bound on Y^{-1}, and minimize $tr(Z)$ instead, i.e, we recast the problem in the following form

Min $tr(Z)$
Subject To LMIs in(29), and

$$\left[\begin{array}{cc} Z & I_{n\times n} \\ I_{n\times n} & Y \end{array}\right] > 0 \quad (30)$$
$$(31)$$

If the above LMIs are feasible, we can calculate the controller gains as

$$K_i = X_i Y^{-1}$$

We demonstrate a numerical example in the next section.

4. Numerical Example

To illustrate this design approach, consider the problem of balancing an inverted pendulum on a cart. We use the same model as in [8]. The equations for the motion of the pendulum are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g\sin(x_1) - amlx_2^2\sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)} \end{aligned} \tag{32}$$

where x_1 denotes the angle of the pendulum (in radians) from the vertical axis, and x_2 is the angular velocity of the pendulum. $g = 9.8m/s^2$ is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, and u is the forced applied to the cart in Newtons. Using the same values as in [8], we have $a = \frac{1}{m+M}$, $m = 2$ kg, $M = 8.0$ kg, and $2l = 1.0$ m in the simulations. We approximate the nonlinear plant by two Takagi-Sugeno fuzzy rules. Note that the plant is not controllable for $X_1 = \pm\pi/2$, therefore we linearize the system around 80° instead . The plant rules are:

Plant Rule (1): If x_1 is *close to zero* **Then** $\dot{x} = A_1x + B_1u$
Plant Rule (2): If x_1 is *close to* $\pm\pi/2$ **Then** $\dot{x} = A_2x + B_2u$
where *close to zero* and *close to* $\pm\pi/2$ are the input fuzzy sets defined by the membership functions $\mu_1 = 1 - \frac{2}{\pi}|x_1|$ and $\mu_2 = \frac{2}{\pi}|x_1|$ respectively. The membership functions are depicted in figure and A_1, A_2, B_1, B_2 are given as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{4l/3-aml} & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -\frac{a}{4l/3-aml} \end{bmatrix} \\ A_2 &= \begin{bmatrix} 0 & 1 \\ \frac{9g}{4\pi(4l/3-aml\beta^2)} & 0 \end{bmatrix} \; B_2 = \begin{bmatrix} 0 \\ -\frac{a\beta}{4l/3-aml\beta^2} \end{bmatrix} \\ \beta &= \cos(80^\circ) \end{aligned} \tag{33}$$

We also assume the following values for Q and R

$$Q = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \tag{34}$$

$$R = 2 \tag{35}$$

Solving the LMI optimization problem in the previous section, we obtain the following values for the controller gains

$$\begin{aligned} K_1 &= \begin{bmatrix} -225.6 & -55.8 \end{bmatrix} \\ K_2 &= \begin{bmatrix} -272.1 & -75.9 \end{bmatrix} \end{aligned} \tag{36}$$

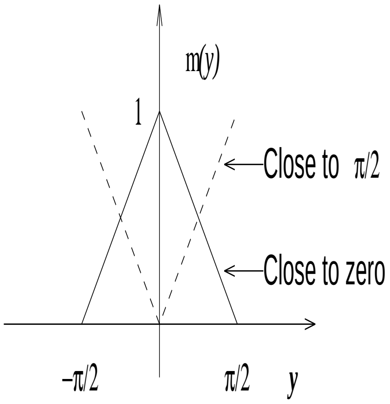

Figure 1: Membership functions for the two fuzzy sets *close to zero* and *close to* $\pm\pi/2$.

The resulting global controller is

$$u = -(\mu_1(x_1)K_1 + \mu_2(x_1)K_2)x$$

Simulations indicate that the above control law can balance the pendulum for initial conditions between [-80 °,80°]. Results are depicted in figures 1-3. As it is evident from the simulation results, the controller gains are much smaller than the ones given in [8]. It is worthwhile to note that we can design nonlinear controllers for the plant based on feedback linearization linearization techniques, but these controllers are usually very complicated. One example of such controllers is given in [8].

5. Conclusion

The purpose of this paper was to extend some recent reults on the synthesis of T-S fuzzy controllers. We presented a controller which minimizes an upper bound of a linear quadratic performance measure using the guaranteed cost approach. The results obtained here can be extended to an LQG scheme, using the observer strategy given in [3, 4].

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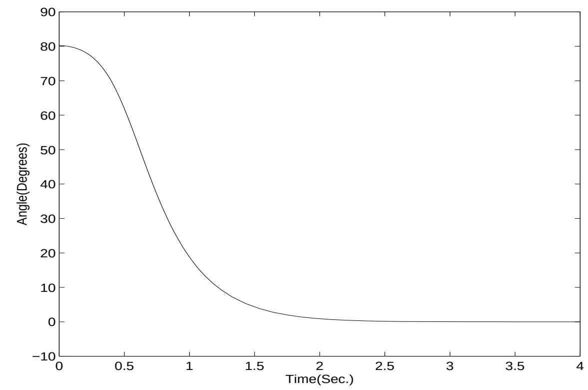

Figure 2: Initial Condition Response of the Pendulum Angle.

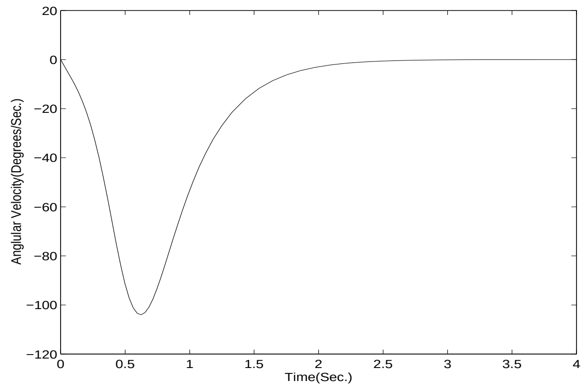

Figure 3: Initial Condition Response of Pendulum's Angular Velocity.

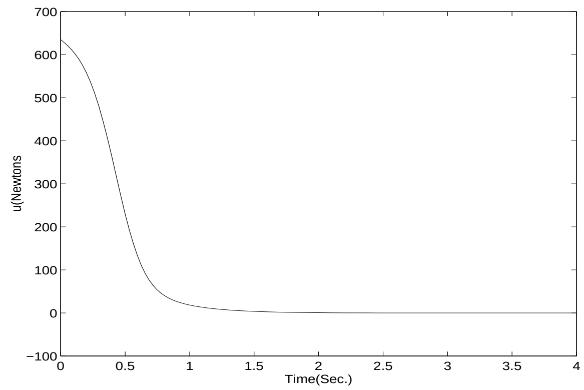

Figure 4: Control Action.

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