1. VMTS Homework #1—due Thursday, July 6

Exercise 1. In triangle ABC, let P be $\frac{2}{3}$ of the way from A to B, let Q be $\frac{1}{3}$ of the way from P to C, and let the line BQ intersect AC in R. How far is R along the way from A to C?

Exercise 2. In triangle ABC, let S divide AB in the ratio $1\colon 3$, let R divide BC in the ratio $1\colon 3$, let P be the intersection of AR and SC, and let BP meet AC in Q. In what ratios does P divide the segments AR, BQ, and CS?

Exercise 3. Consider a tetrahedron $ABCD$ in space. A *median* of this tetrahedron is a line segment from one vertex (say, A) to the centroid of the opposite face (in this case, triangle BCD).

Prove that all four medians of a tetrahedron meet in a single point, which is $\frac{3}{4}$ of the way along each of them. (That point is called the *centroid* of the tetrahedron.)

Exercise 4. **Menelaus' theorem** states that in the following picture:

we have the following equation:

$$\frac{AC'}{C'B}\frac{BA'}{A'C}\frac{CB'}{AB'} = 1. \tag{1}$$

Prove Menelaus' theorem using the techniques described in class.

Exercise 5. The last factor on the left hand side of (1) is usually written $CB'/B'A$, and the product is usually said to be -1. Explain why.

Exercise 6. Using the techniques described in class, prove the **converse of Ceva's theorem**: if A', B', and C' are points on BC, AC, and AB respectively such that

$$(AC')(BA')(CB') = (C'B)(A'C)(B'A)$$

then the segments AA', BB', and CC' all meet in a point P.

2. VMTS Homework #2—due Saturday, July 8

Recall that a **covector** (in 2D) is determined by two parallel lines and a direction from one to the other. Let ω be a covector and $v = \overrightarrow{PQ}$ a polar vector (all in 2 dimensions). Arrange the two so that P is on the initial line of ω and let the line $\overleftrightarrow{PQ}$ intersect the second line of ω at R, as shown:

Define $\omega \cdot v$ to be the ratio PQ/PR.

Exercise 7. Show that:

(1) $\omega \cdot (v + w) = \omega \cdot v + \omega \cdot w$
(2) $\omega \cdot (rv) = r(\omega \cdot v)$
(3) $(\omega + \xi) \cdot v = \omega \cdot v + \xi \cdot v$
(4) $(r\omega) \cdot v = r(\omega \cdot v)$

Exercise 8. What do you think a twisted vector should be in 2 dimensions? (The term 'axial vector' is really appropriate only in 3D.) Define addition and scaling of 2-dimensional twisted vectors.

Exercise 9. Define covectors in 3 dimensions and explain how to add them and pair them with polar vectors.

Exercise 10. Can you define $\omega \cdot v$ if ω is a covector in 3D and v is an *axial* vector?

A function f from polar vectors to real numbers satisfying $f(v+w) = f(v)+f(w)$ and $f(rv) = r(f(v))$ is called a **linear functional**.

Exercise 11. Show that if f is any linear functional (in 2 dimensions), there exists a unique covector ω such that $f(v) = \omega \cdot v$ for all v.

3. VMTS Homework #3—due Tuesday, July 11

Exercise 12. A boat flying a flag is traveling 10 mph due east, while the wind is blowing at 5 mph from north to south. In which direction does the flag point?

Exercise 13. Now suppose that the boat travels in a circle at 5 mph, while the wind remains at 5 mph from north to south. Describe in words the way the direction of the flag changes as the boat moves, (a) relative to the water, and (b) relative to the deck of the boat. *[This is the basis of an astronomical phenomenon known as 'aberration' which causes the apparent position of stars to change as the Earth moves around the sun.]*

Exercise 14. When the wind blows on the sail of a ship, only the component of the wind which is perpendicular to the sail exerts a force on the ship. Similarly, since the boat has a keel, only the component of the force which is parallel to the direction the boat is pointing in contributes to the motion of the boat. Explain how, by holding the sail at the correct angle, a sailing ship can use the force of the wind to move in any direction except directly into the wind.

Exercise 15. *[This question is tricky, and probably requires some calculus.]*

(1) What is the optimal angle at which to hold the sail, for a given wind direction and ship direction?
(2) A ship can also move directly into the wind by zigzagging back and forth at an angle to it. What is the optimal angle the ship should take to the wind?

Exercise 16. Assume that the propellers of a plane are all turning clockwise, as viewed from behind the plane. If the plane turns to the right, what will be the 'gyroscopic' effect of the turning propellers? (Assume that the plane pivots perfectly to the right, i.e. neglect the effect of banking.)

Exercise 17. Can you define the cross product of two axial vectors? How about one polar vector and one axial vector?

4. VMTS Homework #4—due Thursday, July 13

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, whose elements we call *scalars*. Recall that a *vector space* over $\mathbb{K}$ is a set V, whose elements we call *vectors*, together with a way to add vectors and a way to multiply vectors by scalars, such that the following axioms hold.

(1) $u + (v + w) = (u + v) + w$, for all $u, v, w \in V$;
(2) $u + v = v + u$ for all $u, v \in V$;
(3) There is a vector $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
(4) For all $v \in V$ there is a vector $(-v) \in V$ such that $v + (-v) = 0$;
(5) $1v = v$ for all $v \in V$;
(6) $a(bv) = (ab)v$ for all $a, b \in \mathbb{K}$ and $v \in V$;
(7) $(a + b)v = av + bv$ for all $a, b \in \mathbb{K}$ and $v \in V$;
(8) $a(v + w) = av + aw$ for all $a \in \mathbb{K}$ and $v, w \in V$;

Exercise 18. Which of the following are vector spaces over $\mathbb{R}$ (with the obvious definitions of addition and scalar multiplication)? Why or why not?

(1) The set of all polynomials with real coefficients.
(2) The set of all polynomials with integer coefficents.
(3) The set of all polynomials with real coefficients of degree exactly 3.
(4) The set of all polynomials with real coefficients of degree at most 3.
(5) The set of all infinite sequences of real numbers.
(6) The set of all infinite sequences of real numbers, all but finitely many of which are zero.

Exercise 19. Let X be a set, and let

$$FX = \{a_1x_1 + \cdots + a_nx_n : a_i \in \mathbb{K}, x_i \in X\}$$

be the set of all 'formal linear combinations' of elements of X. (The multiplication and plus signs don't denote any actual operation. If we want to be more formal, we can consider such a linear combination to be a function from X to $\mathbb{K}$ which takes nonzero values at only finitely many inputs.) Show that FX is a vector space. If X is a finite set, show that FX is finite-dimensional and its dimension is the cardinality of X.

Exercise 20. Let V be a vector space. A function $f : V \to \mathbb{K}$ is called a *linear functional* on V if it has the following properties:

(1) $f(v + w) = f(v) + f(w)$ for all $v, w \in V$; and
(2) $f(av) = af(v)$ for all $a \in \mathbb{K}$ and $v \in V$.

Let V^* be the set of all linear functionals on V. Show that V^* is a vector space. It is called the *dual space* of V.

Exercise 21. If V is finite-dimensional, show that V^* is also finite-dimensional, and $\dim V^* = \dim V$. In this case, show that V is essentially the same as $(V^*)^*$.

Exercise 22 (*for people who were here last week*)**.** Define the notion of an *affine space* A over $\mathbb{K}$, whose elements we call *points*, using the sort of 'affine combinations' we discussed last week. (This is somewhat tricky to get right, so be careful!) Construct, for any affine space A, a set V_pA of 'polar vectors' in A, and show that it is a vector space.

5. VMTS Homework #5—due Wednesday, July 19

Exercise 23. Let $\mathbb{R}[x]$ denote the vector space of polynomials with real coefficients, and let W be the subspace spanned by the single element $x^2 - 1$. What is the dimension of the quotient space $\mathbb{R}[x]/W$?

Exercise 24. Let V be a finite-dimensional vector space and let $W \subset V$ be a subspace. Show that W is finite-dimensional and that $\dim W \leq \dim V$. Show that if $\dim W = \dim V$, then $W = V$.

Exercise 25. Let V be a finite-dimensional vector space and let $W \subset V$ be a subspace. Show that V/W is finite-dimensional and that

$$\dim V/W = \dim V - \dim W.$$

When is $\dim V/W = \dim V$?

Exercise 26. Show that if V and W are finite-dimensional vector spaces of the same dimension, then $V \cong W$.

Important Note: For the rest of this class (and, in fact, the rest of your life), when you are asked to construct an isomorphism, try to avoid using the type of isomorphism constructed above. Instead, try to find a 'natural' isomorphism which doesn't depend on any arbitrary choices.

Exercise 27. Let $V = \mathbb{R}[x]$ and $W = \mathbb{R}[y]$. Show that $V \otimes W$ is isomorphic to the vector space $\mathbb{R}[x, y]$ of polynomials in two variables.

Let V, W, and Z be vector spaces. A *bilinear map* is a function $B \colon V \times W \to Z$ such that

(1) $B(av, w) = aB(v, w) = B(v, aw)$ for all $v \in V$, $w \in W$, and $a \in \mathbb{K}$;
(2) $B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$ for all $v_1, v_2 \in V$ and $w \in W$; and
(3) $B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2)$ for all $v \in V$ and $w_1, w_2 \in W$.

Exercise 28. Show that there is a canonical bilinear map $\beta \colon V \times W \to V \otimes W$.

Exercise 29. Show that if $B \colon V \times W \to Z$ is any bilinear map, then there is a unique *linear* map $\widetilde{B} \colon V \otimes W \to Z$ such that $B = \widetilde{B} \circ \beta$.

Exercise 30. Let U be a vector space and $\gamma \colon V \times W \to U$ be a bilinear map with the property that if $B \colon V \times W \to Z$ is any bilinear map, then there is a unique linear map $\overline{B} \colon U \to Z$ such that $B = \overline{B} \circ \gamma$. Show that $U \cong V \otimes W$.

Exercise 31. Let U, V, and W be vector spaces. Show that $U \otimes V \cong V \otimes U$ and $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$.

Let V and W be vector spaces and let $B \colon V \times W \to \mathbb{K}$ be a bilinear map to $\mathbb{K}$. We say that B is *nondegenerate* if the following two conditions hold.

(1) If $B(v, w) = 0$ for a fixed $w \in W$ and all $v \in V$, then $w = 0$.
(2) If $B(v, w) = 0$ for a fixed $v \in V$ and all $w \in V$, then $v = 0$.

Exercise 32. Let V and W be finite-dimensional and let $B \colon V \times W \to \mathbb{K}$ be a nondegenerate bilinear map. Show that $V \cong W^*$ and $W \cong V^*$.

6. VMTS Homework #6—due Saturday, July 22

Exercise 33. Explain geometrically why a rotation of the plane acts the same way on a vector no matter what basepoint we use for the vector.

Exercise 34. Let V be a 2-dimensional Euclidean space with a chosen orthonormal basis $\{e_1, e_2\}$. Let $L\colon V \to V$ be a rotation transformation. (Recall that means that it preserves inner products and orientation). Show that there is an angle ϕ such that

$$L(e_1) = (\cos\phi)e_1 + (\sin\phi)e_2$$
$$L(e_2) = (-\sin\phi)e_1 + (\cos\phi)e_2$$

Exercise 35. Let V be a vector space and let $\mathrm{Bilin}(V, \mathbb{K})$ be the set of bilinear maps $V \times V \to \mathbb{K}$. Show that $\mathrm{Bilin}(V, \mathbb{K})$ is a vector space. Then show that if V is finite-dimensional, then $\mathrm{Bilin}(V, \mathbb{K}) \cong V^* \otimes V^*$.

Exercise 36. Let V be a finite-dimensional Euclidean space with a chosen orthonormal basis $\{e_i\}$. (Recall that 'orthonormal' means that $\langle e_i, e_j\rangle$ is 1 if $i = j$ and 0 otherwise.) Show that if $v = v^1 e_1 + \cdots + v^n e_n$, then in fact $v^i = \langle v, e_i\rangle$.

Exercise 37. Let V be a finite-dimensional Euclidean space. Show that V has an orthonormal basis. *(Hint: Think about the 2-dimensional case first. If you start with a basis which is not orthonormal, can you keep the first vector and change the second vector to make them orthogonal? What can you do to make the basis orthonormal?)*

7. VMTS Homework #7—due Thursday, July 27

Exercise 38. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis in a 3-dimensional Euclidean space V. Let $R_{3,\theta}$ be a linear map which rotates by angle θ around the axis e_3. What is a possible matrix of $R_{3,\theta}$ relative to the given basis? (There is more than one possible answer, depending on which direction you rotate.)

Recall that if we have two matrices

$$M = \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_n^m \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} b_1^1 & \dots & b_m^1 \\ \vdots & \ddots & \vdots \\ b_1^p & \dots & b_m^p \end{pmatrix}$$

then their product is a matrix

$$NM = \begin{pmatrix} g_1^1 & \dots & g_n^1 \\ \vdots & \ddots & \vdots \\ g_1^p & \dots & g_n^p \end{pmatrix}$$

where

$$g_i^k = \sum_{j=1}^m a_i^j b_j^k.$$

Exercise 39. Write out a proof that matrix multiplication is associative, using the explicit algebraic definition of matrix multiplication. Observe how unenlightening it is, compared to the proof we gave in class using linear maps. Then tear it up. Do not turn it in.

Recall that the *transpose* of a matrix M (as above) is the matrix

$$M^\top = \begin{pmatrix} a_1^1 & \dots & a_1^m \\ \vdots & \ddots & \vdots \\ a_n^1 & \dots & a_n^m \end{pmatrix}.$$

Exercise 40. Prove that $(NM)^\top = M^\top N^\top$.

Exercise 41. Let V be a finite-dimensional Euclidean space. Prove the Cauchy-Schwartz inequality:

$$\langle v, w\rangle^2 \leq \langle v, v\rangle\langle w, w\rangle.$$

(Hint: compute $\langle v + tw, v + tw\rangle$ and consider it as a polynomial in t. How many real roots can it have? What does that tell you about the coefficients?)

Recall that a matrix is said to be *orthogonal* if $M^\top M = I$, where I is the identity matrix; or equivalently that the columns of M are the coordinates of an orthonormal basis.

Exercise 42. Let V be a finite-dimensional Euclidean space, let $A\colon V \to V$ be a linear map, and let $\{e_i\}$ and $\{e_i'\}$ be two orthonormal bases for V. Show that if the matrix of A with respect to $\{e_i\}$ is orthogonal, then so is the matrix of A with respect to $\{e_i'\}$.

Exercise 43. Let M be an orthogonal matrix. Show that $M^\top$ is also orthogonal, and conclude that $MM^\top = I$ as well.

Exercise 44. Let V be an n-dimensional vector space and let $\mathrm{End}(V)$ be the set of linear maps from V to itself. Observe that $\mathrm{End}(V)$ is a vector space. Prove that $\mathrm{End}(V) \cong V \otimes V^*$. Show that composition of linear maps corresponds to one type of tensor multiplication.

Exercise 45. Show that a linear map $A\colon V \to W$ induces another linear map $A^*\colon W^* \to V^*$. Let $\{e_i\}$ be a basis of V and $\{f^i\}$ the dual basis of V^*. How is the matrix $M(A^*)$ of A^* with respect to the basis $\{f^i\}$ related to the matrix $M(A)$ of A with respect to the basis $\{e_i\}$?

Exercise 46. Can you find a conceptual proof of Exercise 40 using linear maps, analogous to the conceptual proof we gave in class that matrix multiplication is associative? *(Hint: Use the previous exercise.)*

8. VMTS Homework #8—due Saturday, July 29

Let $L\colon V \to V$ be a linear map. Recall that the *determinant* of L is defined to be the scalar $(\det L)$ such that

$$\begin{aligned}\Lambda^n L(e_1 \wedge \cdots \wedge e_n) &= (Le_1) \wedge \cdots \wedge (Le_n)\\ &= (\det L)(e_1 \wedge \cdots \wedge e_n).\end{aligned}$$

Exercise 47. Let V be 3-dimensional and suppose that $L\colon V \to V$ has the following matrix with respect to some basis.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Show that

$$\det L = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

Exercise 48. Derive a general algebraic formula for the determinant of an $n \times n$ matrix:

$$M = \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_n^m \end{pmatrix}$$

Exercise 49. Show that for any $n \times n$ matrix M, we have $\det M = \det\left(M^\top\right)$.

Exercise 50. Show that $\det(MN) = (\det M)(\det N)$ for any $n \times n$ matrices M and N.

A linear map $L\colon V \to V$ is said to be *invertible* if there exists a linear map L^{-1} such that $L \circ L^{-1} = I$ and $L^{-1} \circ L = I$ (where $I\colon V \to V$ is the identity map, $I(v) = v$).

Exercise 51. Show that if L is invertible, then $\det L \neq 0$.

Exercise 52. Show that if $\det L \neq 0$, then L is invertible. *(Hint: recall that if V is n-dimensional, then $v_1 \wedge \cdots \wedge v_n \neq 0$ if and only if $\{v_1, \dots, v_n\}$ is a basis of V.)*

Exercise 53. Show that if L is orthogonal, then $\det L = \pm 1$. Conclude that if L is a rotation transformation, then $\det L = 1$.

Exercise 54. Give an example of a transformation L which is not orthogonal, but such that $\det L = 1$.