Integration by Parts, continued

Remember the product rule:

$$(uv)' = u'v + uv'$$

We can rewrite this expression as

$$uv' = (uv)' - u'v$$

Unlike last lecture, let's take the definate integral of that expression:

$$\int_a^b uv'dx = \int_a^b (uv)'dx - \int_a^b u'vdx$$

By the fundamental theorem of calculus, we can say

$$\int_a^b uv'dx = uv\Big|_a^b - \int_a^b u'vdx$$

Another notation that means the same thing is

$$\int udv = uv\Big| - \int vdu$$

where:

$$dv = v'dx$$

$$uv'dx = udv$$

$$du = u'dx$$

$$u'vdx = vu'dx = vdu$$

Now, let's apply integration by parts to an actual expression.

$$\int (\ln x)dx = ??$$

$$u = \ln x$$

$$du = \frac{1}{x}dx$$

$$dv = dx$$

$$v = x$$

$$\int (\ln x)dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - \int dx = x\ln x - x + c$$

We can also use "advanced guessing" to solve this problem. We know that the derivative of *something* equals $\ln x$:

$$\frac{d}{dx}(??) = \ln x$$

Let's try

$$\frac{d}{dx}(x\ln x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1$$

That's almost it, but not quite. Let's repair this guess to get:

$$\frac{d}{dx}\left(x\ln x - x\right) = \ln x + 1 - 1 = \ln x$$

Reduction Formulas (aka Recurrence Formulas)

$$\int (\ln x)^n \, dx$$

Let's try:

$$u = (\ln x)^n$$

$$u' = n(\ln x)^{n-1}\left(\frac{1}{x}\right)$$

$$v' = dx$$

$$v = x$$

Plugging these into the formula for integration by parts gives us:

$$\int (\ln x)^n dx = x(\ln x)^n - \int n(\ln x)^{n-1} \cancelto{1}{x\left(\frac{1}{x}\right)} dx$$

Keep repeating integration by parts to get the full formula: $n \to n-1 \to n-2 \to n-3 \to$ etc

Here's another example of a reduction formula:

$$\int x^n e^x dx$$

Let's try:

$$u = x^n$$

$$u' = nx^{n-1}$$

$$v' = e^x$$

$$v = e^x$$

Putting these into the integration by parts formula gives us:

$$\int x^n e^x dx = x^n e^x - \int nx^{n-1} e^x dx$$

Repeat, going from $n \to n-1 \to n-2 \to$ etc.

Bad news: If you change the integrals just a little bit, they become impossible to evaluate:

$$\int \left(\tan^{-1} x\right)^2 dx = \text{impossible}$$

$$\int \frac{e^x}{x} dx = \text{also impossible}$$

Good news: When you can't evaluate an integral, then

$$\int_1^2 \frac{e^x}{x}dx$$

is an *answer*, not a question. This *is* the solution– you don't have to integrate it!

The most important thing is setting up the integral! (Once you've done that, you can always just plug it into Mathematica.)

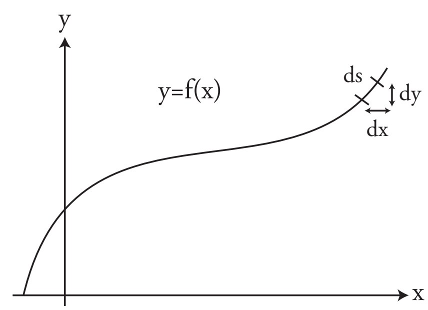
So, why bother to evaluate integrals by hand, then? Because you often get families of related integrals, such as

$$F(a) = \int_1^\infty \frac{e^x}{x^a}dx$$

where you want to find how the answer depends on, say, a.

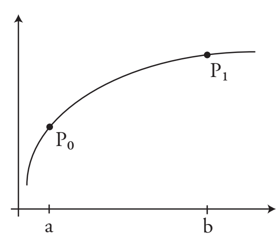
Arc Length

This is very useful to know for 18.02 (multi-variable calc).



Here, S denotes arc length and dS = the infinitesmal of arc length.

$$dS = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$



To find the length of the curve between P_o and P_1, evaluate:

$$\int_{P_o}^{P_1} dS$$

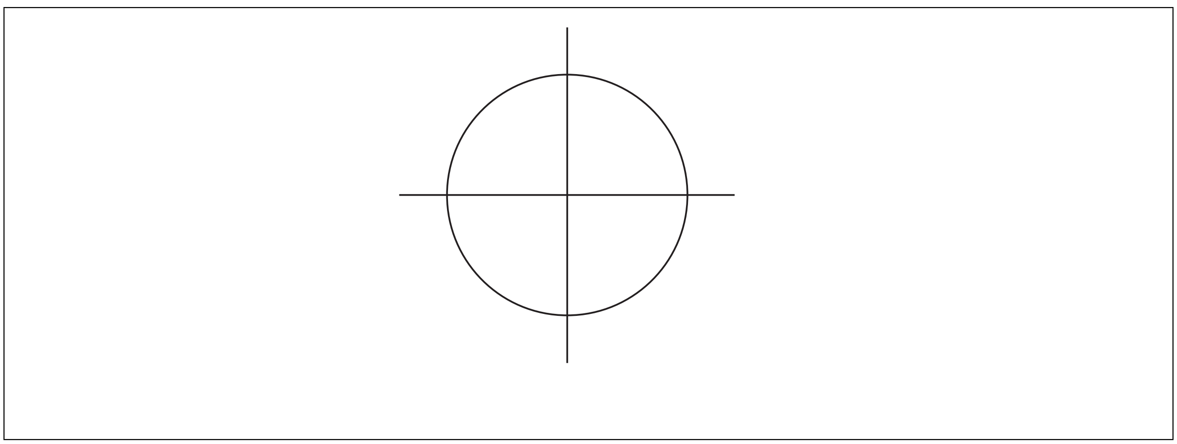
We want to integrate with respect to x, not S, so we do a little algebra to get dS in terms of dx.

$$\frac{(dS)^2}{(dx)^2} = \frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2} = 1 + \left(\frac{dy}{dx}\right)^2$$

Therefore,

$$\int_{P_o}^{P_1} dS = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

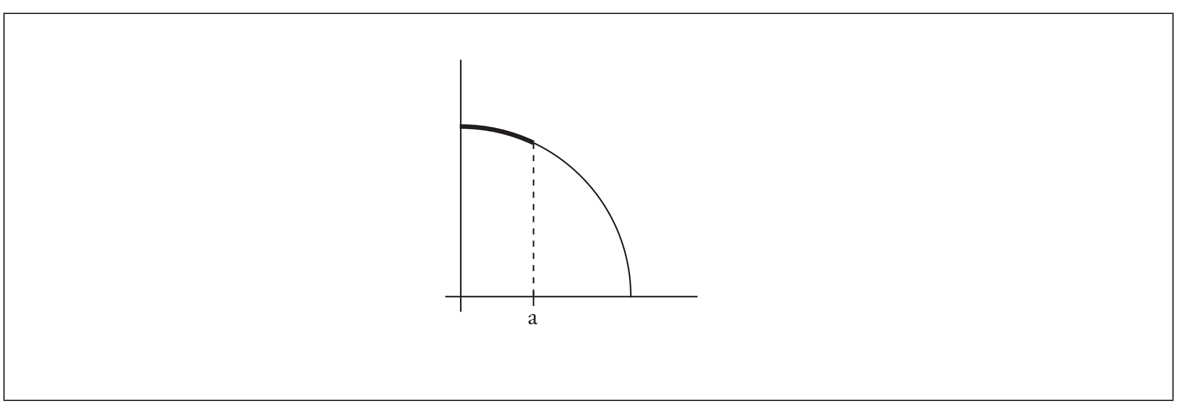
Example 1: the circle



$$x^2 + y^2 = 1$$

$$y = \sqrt{1 - x^2}$$

We want to find the length of this arc:



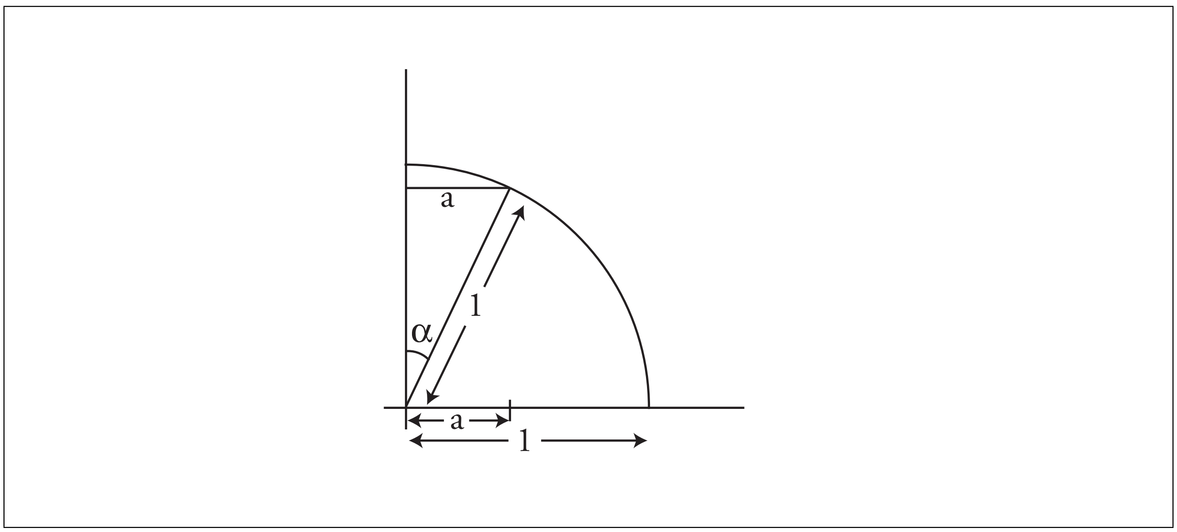
$$\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^2}}\left(\frac{1}{2}\right) = \frac{-x}{\sqrt{1-x^2}}$$

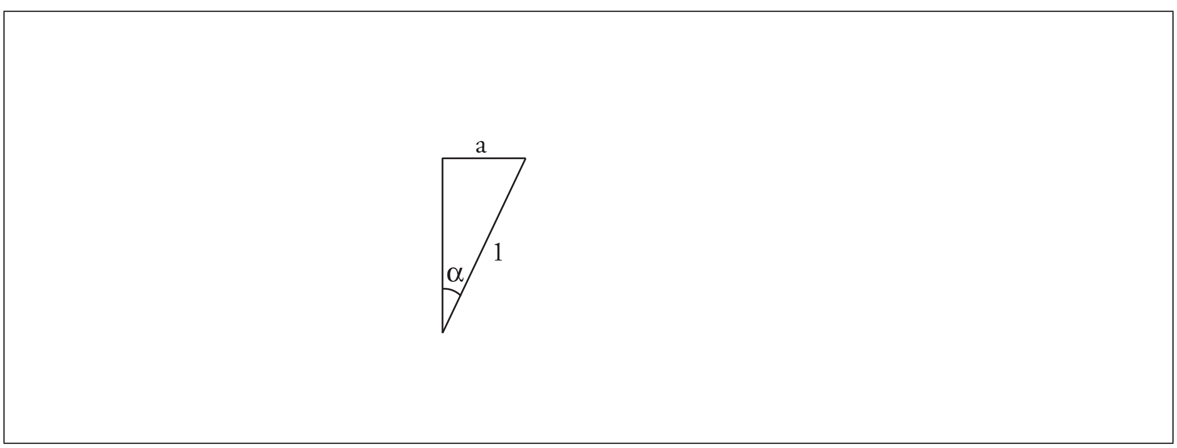
$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx$$

$$1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$dS = \sqrt{\frac{1}{1-x^2}}\,dx$$

$$S = \int_0^a \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\,\Big|_0^a = \sin^{-1} a - \sin^{-1} 0 = \sin^{-1} a$$





Here,

$$\sin\alpha = a$$

Arclength = angle (in radians).

Parametric Equations

Example 1

$$x = a\cos t$$

$$y = a\sin t$$

Ask yourself: what's constant? What's varying? Here, t is variable and a is constant.

Is there a relationship between x and y? Yes:

$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$$

Extra information (besides the circle):
At $t = 0$,

$$x = a\cos 0 = a$$

and

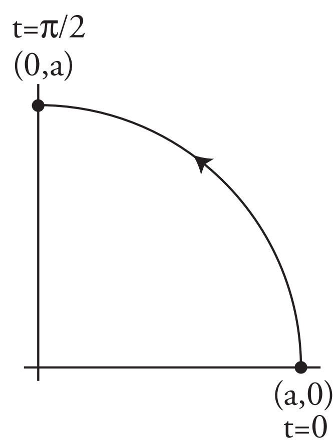
$$y = a\sin 0 = 0$$

At $t = \frac{\pi}{2}$,

$$x = a\cos\frac{\pi}{2} = 0$$

and

$$y = a\sin\frac{\pi}{2} = a$$



The particle is moving counterclockwise.

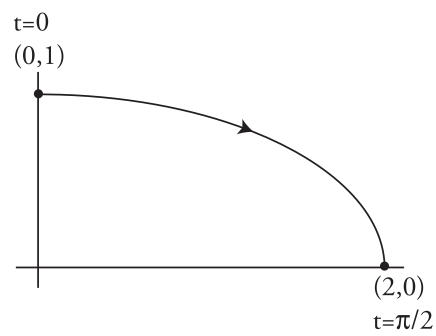
Example 2: ellipse

$$x = 2\sin t$$

$$y = \cos t$$

$$\frac{x^2}{4} + y^2 = 1$$

You can solve the arclength formula in these cases!



From Example 1

$$dS = \sqrt{(a \sin t)^2 + (a \cos t)^2} dt = a dt$$

Most parametric equations can't be integrated like this, however.